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ON THE STABILITY PROPERTIES BY QUASI-EXPECTATION OF STOCHASTIC SET SOLUTIONS WITH SELECTORS

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Abstract. In this paper, we obtain stability properties by quasi-expectation of stochastic set solutions for stochastic set differential equations under Hukuhara derivative (SSDEs) with selectors, for example using different kinds of Lyapunov like functions we have many estimate of stability criteria.

Keywords: Set-valued stochastic processes; selectors; stochastic set control differential equations with selectors; quasi-expectation; stability theory.

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1 Introduction

In probability space $(\Omega, \mathcal{F}, \mathbb{P})$, besides the continuous stochastic processes $x_t = x(t, \omega)$, we are interested in generating the stochastic set of processes $X_t = X(t, \omega) \in K_{CC}(\mathbb{R}^n)$, where $t \in [0, T] \subset R_+$ and $K_{CC}(\mathbb{R}^n)$ are the family of all nonempty convex and compact subsets of Euclidean n-dimensional space $\mathbb{R}^n$ with Hausdorff distance. We can find an extensive literature (see [2–8]), where attempts have been made to investigate stochastic set differential equations (SSDEs) and stochastic differential inclusions (SDIs). In [7] the authors have investigated the problem of existence and uniqueness of stochastic set solutions of SSDEs under classical deriva-

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tive. In [8] the authors have investigated boundedness properties of these solutions of SSDEs under Hukuhara derivative with selectors by many kinds of controls.

In this paper, we have investigated stability properties by quasi-expectation of stochastic set solutions for SSDEs under Hukuhara derivative with selectors without any controls. We organize this paper as follows. In section 2, we introduce some necessary notations, definitions and results about stochastic set differential equations. In section 3 we consider many kinds of stability properties by quasi-expectation.

2 Preliminaries

We recall some notations and concepts presented in detail in recent work of V. Lakshmikantham *et al.* (see [1]). Let $K_{CC}(\mathbb{R}^n)$ denote the collection of all nonempty compact convex subsets of $\mathbb{R}^n$. Given $A, B \in K_{CC}(\mathbb{R}^n)$, the Hausdorff distance between A and B is defined by

$$d_H(A,B) = \max\Big\{\sup_{a\in A}\inf_{b\in B} \|a-b\|\,,\, \sup_{b\in B}\inf_{a\in A} \|b-a\|\Big\} \tag{2.1}$$

and $\{\theta^n\}$ – the zero points set in $K_{CC}(\mathbb{R}^n)$. It is known that $(K_{CC}(R^n), d_H)$ is a complete metric space and $K_{CC}(\mathbb{R}^n)$ is a complete and separable with respect to d_H .

We define the magnitude of a nonempty subset A as

$$d_H(A,\theta^n) = \|A\| = \sup\{\|a\|, a \in A\} \tag{2.2}$$

The Hausdorff metric (2.1) satisfies the properties below:

$$\begin{aligned} d_H(A+C,B+C) &= d_H(A,B) \quad \text{and} \quad d_H(A,B) = d_H(B,A),\\ d_H(\lambda A, \lambda B) &= \lambda\, d_H(A,B),\\ d_H(A,B) &\le d_H(A,C) + d_H(C,B),\\ d_H(A+C,B+D) &\le d_H(A,B) + d_H(C,D) \end{aligned}$$

for all $A, B, C, D \in K_{CC}(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}_+$. If $\alpha, \beta \in \mathbb{R}$ and $A, B \in K_{CC}(\mathbb{R}^n)$, then

$$\alpha(A+B) = \alpha A + \alpha B, \quad \alpha(\beta A) = (\alpha\beta)A, \quad 1\cdot A = A\,. \tag{2.3}$$

Let $A, B \in K_{CC}(\mathbb{R}^n)$. The set $C \in K_{CC}(\mathbb{R}^n)$ satisfying $A = B + C$ is called the Hausdorff difference (the geometric difference) of the sets A and B and is denoted by $A - B$. Given an interval I in $\mathbb{R}_+$, we say that the set stochastic mapping $X : I \times \Omega \to K_{CC}(\mathbb{R}^n)$ has a Hukuhara derivative $D_H X(t_0, \cdot)$ at a point $t_0 \in I$, if

$$\lim_{h\to 0^+} \frac{X(t_0+h,\cdot) - X(t_0,\cdot)}{h} \quad \text{and} \quad \lim_{h\to 0^+} \frac{X(t_0,\cdot) - X(t_0-h,\cdot)}{h} \tag{2.4}$$

exist in the topology of $K_{CC}(\mathbb{R}^n)$ and are equal, to what we then call $D_H X(t_0, \cdot)$.

By embedding $K_{CC}(\mathbb{R}^n)$ as a complete cone in a corresponding Banach space and taking into account results about the differentiation of Bochner integrals, we find that if

$$X(t,\cdot) = X_0 + \int_{t_0}^{t} \Phi(s,\cdot)\,\mathrm{d}s, \quad X_0 : \Omega \to K_{CC}(\mathbb{R}^n), \tag{2.5}$$

where $\Phi : I \times \Omega \to K_c(\mathbb{R}^n)$ is integrable in the sense of Bochner, then $D_H X(t,\cdot)$ exists and the equality $D_H X(t,\cdot) = \Phi(t,\cdot)$ a.e on $I \times \Omega$ holds.

The Hukuhara integral of X is given by

$$\int_I X(s,\cdot)\,\mathrm{d}s = cl\Big[\int_I x(s,\cdot)\,\mathrm{d}s : x \text{ is a continuous selector of } X\Big]$$

for any compact set $I \subset \mathbb{R}_+$.

Some properties of the Hukuhara integral are given in [8]. If $X : I \times \Omega \to K_{CC}(\mathbb{R}^n)$ is integrable, one has

$$\int_{t_0}^{t_2} X(s,\cdot)\,\mathrm{d}s = \int_{t_0}^{t_1} X(s,\cdot)\,\mathrm{d}s + \int_{t_1}^{t_2} X(s,\cdot)\,\mathrm{d}s, \quad t_0 \le t_1 \le t_2$$

and

$$\int_{t_0}^{t} \lambda X(s,\cdot)\,\mathrm{d}s = \lambda \int_{t_0}^{t} X(s,\cdot)\,\mathrm{d}s, \quad \lambda \in \mathbb{R}.$$

If $X_1, X_2 : I \times \Omega \to K_{CC}(\mathbb{R}^n)$ are integrable, then $d_H(X_1(\cdot,\cdot), X_2(\cdot,\cdot)) : I \times \Omega \to \mathbb{R}$ is integrable and

$$d_H\Big(\int_{t_0}^{t} X_1(s,\cdot)\,\mathrm{d}s, \int_{t_0}^{t} X_2(s,\cdot)\,\mathrm{d}s\Big) \le \int_{t_0}^{t} d_H\Big(X_1(s,\cdot), X_2(s,\cdot)\Big)\,\mathrm{d}s.$$

In [9], the authors considered stochastic differential equations (SDEs)

$$\mathrm{d}x_t = f(t,x_t)\,\mathrm{d}t + g(t,x_t)\,\mathrm{d}w_t \tag{2.6}$$

and stochastic differential inclusions (SDIs), see [2, 3],

$$\mathrm{d}x_t \in F(t,x_t)\,\mathrm{d}t + G(t,x_t)\,\mathrm{d}w_t \tag{2.7}$$

where x_t is a stochastic process, $F(t,x_t)$, $G(t,x_t)$ are mappings of set-valued stochastic processes, w_t is the Wiener process. Given a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ satisfying the usual conditions, *i.e.*, $\{\mathcal{F}_t\}_{t\geqslant 0}$ is an increasing and right continuous family of σ-subalgebras of $\mathcal{F}$ and $\mathcal{F}_0$ contains all $\mathbb{P}$–null sets. Let $\mathcal{P}(\mathcal{F}_t)$ denote the smallest σ-algebra on $\mathbb{R}_+ \times \Omega$ with respect to which every continuous adapted process is measurable. An n-dimensional stochastic process X is said to be predictable (progressively measurable, nonanticipating) if X is $\mathcal{P}(\mathcal{F}_t)$-measurable. Let $w(t), t \in [0,T]$ be an $\mathcal{F}_t$–adapted one-dimensional Wiener process defined on $(\Omega, \mathcal{F}, \mathbb{P})$. We shall deal with measurable multifunctions defined on Ω with values in the family of nonempty closed subsets of $K_{CC}(\mathbb{R}^n)$. In [7] the authors have investigated the stochastic set differential equation (SSDE) under classical derivative:

$$\mathrm{d}X(t) = F(t,X(t))\,\mathrm{d}t + G(t,X(t))\,\mathrm{d}w(t) \tag{2.8}$$

Denote by $\mathcal{M}(\Omega, \mathcal{F}_t, \mathbb{P}; K_{CC}(\mathbb{R}^n))$ the family of these mappings of set-valued stochastic processes, whose range of the image belongs to $K_{CC}(\mathbb{R}^n)$. In [8] the authors have considered stochastic set differential equations (SSCDEs) under Hukuhara derivative with selectors:

$$D_H X_t = F\left(t, X_t, U_t\right) + G\left(t, X_t, U_t\right)\xi(t) \tag{2.9}$$

where $X_t = X(t,\omega) \in \mathcal{M}(\Omega, \mathcal{F}_t, \mathbb{P}; K_{CC}(\mathbb{R}^n))$, $t \in [0,T] \subset \mathbb{R}^+$, $\xi(\cdot)$ is one-dimensional "*white noise*", *i.e.*, the time derivative of the Wiener process, and $F, G \in \mathcal{M}(\Omega, \mathcal{F}_t, \mathbb{P}; K_{CC}(\mathbb{R}^n))$ such that:

$$F : [0,T] \subset \mathbb{R}^+ \times \Omega \times K_{CC}(\mathbb{R}^n) \times K_{CC}\left(\mathbb{R}^d\right) \to K_{CC}(\mathbb{R}^n),$$
$$F(t, X_t, U_t) = F(t, \omega, X(t,\omega), U(t,\omega))$$

is measurable and *Aumann* integrably bounded $\phi_t(\omega) = \int\limits_0^t F(s, X_s, U_s)\,\mathrm{d}s$.

$$G : [0,T] \subset \mathbb{R}^+ \times \Omega \times K_{CC}(\mathbb{R}^n) \times K_{CC}\left(\mathbb{R}^d\right) \to K_{CC}(\mathbb{R}^n),$$
$$G(t, X_t, U_t) = G(t, \omega, X(t,\omega), U(t,\omega))$$

is measurable and *Itô* integrably bounded, $I_t(\omega) = \int\limits_0^t G(s, X_s, U_s)\,\mathrm{d}w_s$, where w_t is an $\{\mathcal{F}_t\}$–adapted one-dimensional Wiener process with $\dfrac{\mathrm{d}w_t}{\mathrm{d}t} = \xi(t)$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 2.1 *A set-valued stochastic process* $X_t = X(t,\omega) \in \mathcal{M}(\Omega, \mathcal{F}_t, \mathbb{P}; K_{CC}(\mathbb{R}^n))$*, with* $\omega \in \Omega$*,* $t \in [0,T] \subset \mathbb{R}^+$*, is called stochastic set solution to SSDE* (2.8)*, if it satisfies*

(i) X_t *is a continuous mapping with respect to the metric* d_H*.*

(ii) for every $t \in [0,T]$*,*

$$X_t = X_0 + \int\limits_0^t F(s, X_s)\,\mathrm{d}s + \int\limits_0^t G(s, X_s)\,\mathrm{d}w_s \quad \mathbb{P}\text{–a.e.}. \tag{2.10}$$

where $X_0 : \Omega \to K_{CC}(\mathbb{R}^n)$ *is an* $\mathcal{F}_0$*-measurable multifunction.*

Definition 2.2 *(see [7]) A stochastic set solution* $X_t : [0,T] \times \Omega \to K_{CC}(\mathbb{R}^n)$ *of SSDE* (2.8) *is unique if for every* $t \in [0,T]$

$$X_t = Y_t \qquad \mathbb{P}\text{–a.e.}$$

where $Y_t : [0,T] \times \Omega \to K_{CC}(\mathbb{R}^n)$ *is any set solution of* (2.8)*.*

Assume that $F, G : [0,T] \times \Omega \times K_{CC}(\mathbb{R}^n) \to K_{CC}(\mathbb{R}^n)$ satisfy the following conditions:

(FG1) For every set $A \in K_{CC}(\mathbb{R}^n)$ the mappings $F(\cdot,\cdot,A), G(\cdot,\cdot,A)$ are nonanticipating multifunctions.

(FG2) There exists a constant $L > 0$, such that

$$\max\{d_H(F(t,\omega,A), F(t,\omega,B)), d_H(G(t,\omega,A), G(t,\omega,B))\} \le L \cdot d_H(A,B).$$

(FG3) There exists a constant $C > 0$, such that

$$\max\{d_H(F(t,\omega,A), \{\theta^n\}), d_H(G(t,\omega,A), \{\theta^n\})\} \le C \cdot (1 + d_H(A, \{\theta^n\})).$$

Theorem 2.3 *(see [7]) Let* $X_0 \in L^2(\Omega, \mathcal{F}, \mathbb{P}, K_{CC}(\mathbb{R}^n))$ *be an* $\mathcal{F}_0$ *measurable multifunction. If* F, G *satisfy (FG1)–(FG3), then SSDE (2.8) has a unique solution* X_t*.*

Definition 2.4 *Let $X_t, Y_t \in K_{CC}(\mathbb{R}^n)$ be set solutions of (2.9). We have:*

(a) The distance between two stochastic set processes $X_t, Y_t \in K_{CC}(\mathbb{R}^n)$,

$$d_H(X_t, Y_t) = \max\Big\{ \sup_{x_t \in X_t} \inf_{y_t \in Y_t} \|x_t - y_t\|, \sup_{y_t \in Y_t} \inf_{x_t \in X_t} \|y_t - x_t\| \Big\}. \tag{2.11}$$

(b) The norm of a stochastic set process X_t,

$$\|X_t\| = d_H(X_t, \{\theta^n\}) = r(t). \tag{2.12}$$

(c) The p-exponential norm of a stochastic set process X_t,

$$\|X_t\|^p = d_H^p(X_t, \{\theta^n\}) = r^p(t). \tag{2.13}$$

(d) The p-quasi-expectation of a stochastic set process X_t is defined as

$$E_T(\|X_t\|^p) = \int_0^T \|X_t\|^p \, \mathrm{d}t. \tag{2.14}$$

(e) The stochastic set processes $X_t = Y_t$ if and only if

$$E_T(\|X_t\|^p) = E_T(\|Y_t\|^p) \quad a.e.. \tag{2.15}$$

3 Main results

Let's consider the stochastic set differential equations (SSDEs) under Hukuhara derivative and with selectors,

$$D_H X_t = F(t, X_t) + G(t, X_t)\xi(t) \tag{3.1}$$

where $X_t = X(t, \omega) \in K_{CC}(\mathbb{R}^n), t \in J \subset \mathbb{R}^+, \omega \in \Omega$ and $F(t, X_t)$, $G(t, X_t)$ are mappings of set-valued stochastic processes with selectors

$$F(t, X_t) = cl\left\{f^i(t, x_t)\right\} \text{ and } G(t, X_t) = cl\left\{g^i(t, x_t)\right\}, \ i \in I \subset \mathbb{N}. \tag{3.2}$$

Equations (3.1) with (3.2) are the symbolic representation of the following integral expression:

$$X_t = X_0 + \int_0^t F(s, X_s) \, \mathrm{d}s + \int_0^t G(s, X_s) \, \mathrm{d}w_s \tag{3.3}$$

Definition 3.1 *Let F_1, F_2 be mappings of set-valued stochastic processes with selectors. We define the distance $d_H^p(F_1, F_2)$ between these mappings as*

$$d_H^p(F_1, F_2) = \max\Big\{ \sup_i \inf_j \left\| f_1^i(t, \omega, x_t) - f_2^j(t, \omega, y_t) \right\|^p, \sup_j \inf_i \left\| f_2^j(t, \omega, y_t) - f_1^i(t, \omega, x_t) \right\|^p \Big\}$$

where $(t, \omega) \in J \times \Omega$ and $i, j \in I \subset \mathbb{N}$.

Definition 3.2 *A set-valued stochastic process with selectors is called stochastic set solution to the SSDE (3.1) with selectors if it satisfies* (3.1)–(3.3)

$$X_t = cl\left\{x_t^i = x^i(t,\omega) = \left(x_1^i(t,\omega), x_2^i(t,\omega), ..., x_n^i(t,\omega)\right) \mid i \in I \subset \mathbb{N}\right\},$$

and

$$d_H^p\left(X_t, \{\theta^n\}\right) = \|X_t\|^p = \sup\left\{\left\|x_t^i\right\|^p \mid x_t^i \in \mathbb{R}^n \mid i \in I \subset \mathbb{N}\right\}.$$

Definition 3.3 *The family of all $\mathcal{L}^p$-selectors $\left\{x_t^i\right\}$ is denoted*

$$S^p(X) = \left\{x^i(t,\omega) \in \mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{R}^n) \mid i \in I \subset \mathbb{N},\, x_t^i \in X_t \in K_{CC}(\mathbb{R}^n)\right\}.$$

Remark 3.4 *Let $X_t \in K_{CC}(\mathbb{R}^n)$. If $S^p(X) \neq \emptyset$, then there exists a sequence $\left\{x_k^i(t,\omega)\right\}_{k\in\mathbb{N}}$ contained in $S^p(X)$ such that $X_t = cl\left\{x_k^i(t,\omega) \mid i \in I \subset \mathbb{N}, k \in \mathbb{N}\right\}$.*

Remark 3.5 *If $S^p(X) = S^p(Y) \neq \emptyset$, then $X_t = Y_t$ a.e., for all $X_t, Y_t \in K_{CC}(\mathbb{R}^n)$.*

Definition 3.6 *Let $X_t \in K_{CC}(\mathbb{R}^n)$. We define a set-valued mapping $L_t(X)$*

$$\forall (t,\omega) \in J \times \Omega: \;\; L_t(X)(\omega) = \int_0^t X(s,\omega)\,\mathrm{d}s.$$

Lemma 3.7 *(see [8], Theorem 3.9) Let $X = \{X_t \in K_{CC}(\mathbb{R}^n),\, t \in J\}$. Then there exists a sequence of selectors-stochastic processes $\left\{x_k^i(t,\omega) \mid i \in I \subset \mathbb{N},\, k \in \mathbb{N}\right\} \subseteq S^p(X)$ such that:*

(a) $X_t = cl\left\{x^i(t,\omega) \mid i \in I \subset \mathbb{N}\right\}$, *a.e.* $(t,\omega) \in J \times \Omega$,

(b) $L_t(X)(\omega) = cl\left\{\int_0^t x^i(s,\omega)\,\mathrm{d}s \mid i \in I \subset \mathbb{N}\right\}$, *a.e.* $(t,\omega) \in J \times \Omega$,

(c) $L_t(X)(\omega) = cl\left\{L_{t_1}(X)(\omega) + \int_{t_1}^t X(s,\omega)\,\mathrm{d}s\right\}$, *a.e.* $(t,\omega) \in J \times \Omega$,

where the closure is taken in $\mathbb{R}^n$.

Definition 3.8 *Let $X_t \in K_{CC}(\mathbb{R}^n)$, then for all $t \in J$ we define:*

(i) $L_t^p(X)(\omega) = cl\left\{\int_0^t \left\|x^i(s,\omega)\right\|^p \mathrm{d}s \mid i \in I \subset \mathbb{N}\right\}$.

(ii) $E_t\left(\|X_s\|^p\right) = \int_0^t \|X_s\|^p\,\mathrm{d}s = \int_0^t r^p(s)\,\mathrm{d}s$,

Lemma 3.9 *(see [8], Lemma 3.11) Let $X_t, Y_t \in K_{CC}(\mathbb{R}^n)$. Then there exists a measurable subset $A \subseteq [0,T] \times \Omega$ with $(\lambda \times \mu)(A) = 0$ so that the following holds:*

(a) $L_t^p(X)(\omega) \subseteq cl\left\{L_{t_1}^p(X)(\omega) + \int_{t_1}^{t} d_H^p(X_t, \{\theta^n\})\,\mathrm{d}s\right\}$.

(b) *There exists* $\gamma(t) \in \mathbb{R}$ *such that* $d_H^p(L_t(X)(\omega), L_t(Y)(\omega)) \leqslant \gamma(t)\, E_t(d_H^p(X, Y))$.

where the closure is taken in $\mathbb{R}^n$.

Lemma 3.10 *(see [8], Lemma 3.12) If* $X_t = cl\left\{x_t^i \in S^p(X)\right\}$, *then*

(a) $E_t(\|X_s\|^p) \geqslant L_t^p(X)(\omega)$.

(b) $E_T(\|X_s\|^p) \geqslant E_t(\|X_s\|^p)$, *for* $t \in [0, T]$.

(c) $E_t(\|X_s\|^p) \geqslant E_t\left(\left\|x_s^i\right\|^p\right)$.

Now we investigate the some kinds of stability properties by Quasi-expectation (stability by Lyapunov's means (p-LS), p-equi-stability for mappings (p-S), ..., stability for mappings (p-LSM), p-Lyapunov's stability for selectors (p-LSS).

Definition 3.11 *The trivial stochastic set solution of* (3.1) *is said to be*

(p-LS) *p-Lyapunov stable for mapping, if for each* $\varepsilon_{mp} > 0$, *there is* $t_0 > 0$ *and* $\eta_{mp} = \eta_{mp}(t_0, \varepsilon_{mp})$, *such that* $E_{t_0}(\|X_t\|^p) \leq \eta_{mp}$ *implies* $\forall T \geq t_0 :\ E_T(\|X_t\|^p) < \varepsilon_{mp}$.

(p-ALS) *p-Asymptotical Lyapunov stable for mapping, if it is (p-LS) and* $\lim\limits_{T \to +\infty} E_T(\|X_t\|^p) = 0$.

(p-ELS) *p-Exponent Lyapunov stable for mapping, if there exist* $\alpha, \beta > 0$, *such that:*

$$E_T(\|X_t\|^p) \leq \beta \cdot E_{t_0}(\|X_t\|^p) \cdot \exp\left[-\alpha(T - t_0)\right] .$$

Theorem 3.12 *Suppose that the positive Lyapunov-like function* $V(t, X_t) \in C([0, T] \times K_{CC}(\mathbb{R}^n), \mathbb{R}^+)$ *satisfies the following conditions :*

(i) $\left|V(t, X_t) - V\left(t, \overline{X}_t\right)\right| \leqslant L \cdot E_t\Big(d_H^p(X_t, \overline{X}_t)\Big)$, *where* $L > 0$ *is Lipschitz constant for all* X_t, $\overline{X}_t \in K_{CC}(\mathbb{R}^n)$ *and* $t \in [0, T]$.

(ii) The Dini derivative satisfies $D^+V(t, X_t) \leqslant g(t, V(t, X_t))$, *where* $g(t, 0) = 0$, $g \in C\left(\left[\mathbb{R}_+^2, \mathbb{R}\right]\right)$, *and*

$$D^+V(t, X_t) = \lim_{h \to 0} \inf \frac{1}{h}\left[V\left(t + h, X_t + h\left(F(t, X_t) + G(t, X_t)\xi(t)\right)\right) - V(t, X_t)\right] .$$

If X_t *is a stochastic set solution of SSDE* (3.1), *then* $V(t, X_t) \leqslant l(t, t_0, k(t_0))$, *where* $l(t, t_0, k(t_0))$ *is a maximal solution of the ordinary differential equation (ODE)*

$$\frac{\mathrm{d}l}{\mathrm{d}t} = g(t, k(t)),\ \ k(t_0) = k_0 > 0.$$

Proof. Let $X_t = X(t, \omega, t_0, X_0)$ be any stochastic set solution of SSDE (3.1) existing on $[t_0, T] \subset \mathbb{R}^+$. Define the function $m(t) = V(t, X_t)$ so that $m(t_0) = V(t_0, X_{t_0}) \leqslant k_0$. Now for small $h > 0$, by our assumption it follows that

$$m(t+h) - m(t) = V(t+h, X_{t+h}) - V(t, X_t) .$$

Then we have $\forall t \geqslant t_0$, $D^+m(t) = D^+V(t, X_t) \leqslant g(t, m(t))$, where $m(t_0) \leqslant k_0$, and the estimate $\forall t \geqslant t_0, m(t) \leqslant l(t, t_0, k_0)$. □

Corollary 3.13 *If the Lyapunov-like function $V(t, X_t) \in C([0,T] \times K_{CC}(\mathbb{R}^n), \mathbb{R}^+)$ satisfies the conditions in Theorem 3.12, then we have the estimate:*

$$\forall t \geqslant t_0 > 0,\ V(t, X_t) \leqslant V(t_0, X_{t_0}) .$$

Theorem 3.14 *Assume that F, G satisfy (FG1)–(FG3), there is a positive Lyapunov function $V(t, X_t) \in C([0,T] \times K_{CC}(\mathbb{R}^n), \mathbb{R}^+)$ which satisfies the conditions of Theorem 3.12, and there are $a, b \in \mathbb{R}^+$ such that $b\, E_T(\|X_t\|^p) \leq V(t, X_t) \leq a\, E_T(\|X_t\|^p)$. Then:*

(a) If $g(t, V(t, X_t)) \leq 0$, then the trivial stochastic set solution is (p-LS).

(b) If $g(t, V(t, X_t)) < 0$, then the trivial stochastic set solution is (p-ALS).

(c) If $g(t, V(t, X_t)) \leq -\alpha\, V(t, X_t)$, then the trivial stochastic set solution is (p-ELS).

Proof. (a) If F, G satisfy (FG1)–(FG3) then SSDE (3.1) has a unique set solution X_t (3.3). Next, we have to prove that:

$$\forall \varepsilon_{mp} > 0, \exists \eta_{mp}(t_0, \varepsilon_{mp}) : E_{t_0}(\|X_t\|^p) < \eta_{mp} \to E_T(\|X_t\|^p) < \varepsilon_{mp}.$$

Because $D^+V(t, X_t) \leq g(t, V(t, X_t)) \leq 0$, then using Corollary 3.13 we have $\forall t \geq t_0$: $V(t, X_t) \leq V(t_0, X_{t_0})$. By assumptions of this Theorem, we infer

$$b\, E_T(\|X_t\|^p) \leq V(t, X_t) \leq V(t_0, X_{t_0}) \leq a\, E_{t_0}(\|X_t\|^p)$$

where $T \geq t > t_0$. Choosing $\eta_{mp} = \frac{b}{a}\varepsilon_{mp}$, we have $E_T(\|X_t\|^p) < \varepsilon_{mp}$.

(b) Suppose that $g(t, V(t, X_t)) < 0$. Then the trivial stochastic set solution is (p-LS). We have to prove that $\lim\limits_{T\to\infty} E_T(\|X_t\|^p) = 0$. If it is not true, take ε_0 and $\eta > 0$ such that $E_T(\|X_t\|^p) > \eta$ and $a\,\varepsilon_0 \leq b\,\eta$.

On the other hand, we have

$$b\,\eta < b\, E_T(\|X_t\|^p) \leq V(t, X_t) \leq a\ \varepsilon_0 \leq b\,\eta$$

This contradiction proves (p-ALS).

(c) Assume that $g(t, V(t, X_t)) \leq -\alpha\, V(t, X_t)$, that means $D^+V \leq -\alpha V(t, X_t)$ and implies

$$\forall t > t_0 : V(t, X_t) \leq V(t_0, X_{t_0}) \cdot \exp[-\alpha(t - t_0)] .$$

Thus, we have $b\, E_T(\|X_t\|^p) \leq V(t, X_t) \leq V(t_0, X_{t_0}) \cdot \exp[-\alpha(t - t_0)] \leq a\, E_T(\|X_t\|^p) \cdot \exp[-\alpha(t - t_0)]$ or $E_T(\|X_t\|^p) \leq \beta\, E_T(\|X_t\|^p) \cdot \exp[-\alpha(t - t_0)]$, where $\beta = \dfrac{a}{b}$. The proof is completed.

□

Definition 3.15 *The trivial stochastic set solution of SSDE* (3.1) *is said to be:*

(p-S1) *p-equi-stable, if for each* $\varepsilon > 0$ *and* $t_0 > 0$*, there exists* $\eta = \eta(t_0, \varepsilon)$ *such that* $E_{t_0}(\|X_s\|^p) < \eta$ *implies that* $E_t(\|X_s\|^p) < \varepsilon$.

(p-S2) *p-uniformly stable, if* η *in (p-S1) is independent of* T_0.

(p-S3) *p-quasi-equi-asymptotically stable, if for each* $\varepsilon > 0, t_0 > 0$*, there exist* $t' = t'(t_0, \varepsilon)$ *and* $\eta_0 = \eta_0(t_0)$ *such that* $E_{t_0}(\|X_t\|^p) < \eta_0$ *implies that* $\forall T \geq t' + t_0 > 0,\ E_T(\|X_t\|^p) < \varepsilon$.

(p-S4) *p-quasi-uniformly-asymptotically stable, if* η_0 *and* t' *in (p-S3) are independent of* t_0.

(p-S5) *p-equi-asymptotically stable, if (p-S1) and (p-S3) hold simultaneously.*

(p-S6) *p-uniformly asymptotically stable, if (p-S2) and (p-S4) hold simultaneously.*

(p-S7) *p-exponent-asymptotically stable, if there are* $\alpha, \beta > 0$ *such that:*

$$E_T(\|X_s\|^p) \leq \beta\, E_{t_0}(\|X_s\|^p) \cdot \exp\left[-\alpha\,(T - t_0)\right].$$

Remark 3.16 *According to Definitions 3.11 and 3.15, we can say that*

(i) The stochastic set solution of SSDE (3.1) *is (p-S1) if and only if it is (p-LS),* i.e., *(p-S1)* $\Leftrightarrow$ *(p-LS).*

(ii) (p-S6) $\Leftrightarrow$ *(p-ALS).*

(iii) (p-S7) $\Leftrightarrow$ *(p-ELS).*

(iv) (p-S6) or (p-ALS) $\Rightarrow$ *(p-S6).*

(v) (p-S6)$\Rightarrow$ *(p-S4)*

Thus we have to prove (p-S1), (p-S6) and (p-S7).

Theorem 3.17 *Suppose that the positive Lyapunov-like function* $V(t, X_t)$ *satisfies:*

(i) $\left|V(t, X_t) - V\left(t, \bar{X}_t\right)\right| < L \cdot E_T(d_H^p(X_t, \bar{X}_t))$ *where* $L > 0$ *is Lipschitz constant for all* X_t*,* $\overline{X_t} \in K_{CC}(\mathbb{R}^n)$ *and* $t \in [0, T]$.

(ii) There exist $a, b \in \mathbb{R}^+$ *such that:* $b\, E_t(\|X_t\|^p) \leq V(t, X_t) \leq a\, E_t(\|X_t\|^p)$.

(iii) $D^+V(t, X_t) \leq g(t, V(t, X_t))$.

Then:

(a) If $g(t, V(t, X_t)) \leq 0$*, then (p-S1) holds.*

(b) If $g(t, V(t, X_t)) < 0$*, then (p-S6) holds.*

(c) If $g(t, V(t, X_t)) < -\alpha\, V(t, X_t)$*, then (p-S7) holds.*

Proof. We prove anologous as in Theorem 3.14 with the affirmation for (p-S1), (p-S6) and (p-S7) are proved analogous proof of the affirmations for (p-LS), (p-ALS), (p-ELS), respectively. □

Definition 3.18 *Let $X_t \in K_{CC}(\mathbb{R}^n)$ be defined by equation (3.3). Then*

(i) The family of all initial selectors $\left\{x_{t_0}^i\right\}$ is denoted by the sets

$$\widehat{S}_0(X) = \left\{x_0^i(t_0,\omega) \mid x_0^i \in X_0 \in K_{CC}(\mathbb{R}^n),\, i \in I \subset \mathbb{N}\right\}$$

(ii) The family of all solutions selectors $\left\{x_t^i\right\}$ is denoted by the sets

$$\widehat{S}(X) = \left\{x_t^i(t,\omega) \mid x_t^i \in X_t \in K_{CC}(\mathbb{R}^n),\, i \in I \subset \mathbb{N}\right\}$$

Definition 3.19 *Let $\widehat{S}(X)$, $\widehat{S}(Y)$ be as in Definition (3.18). Then, we define:*

(i) The distance between these two families of selectors,

$$d_H^p\left(\widehat{S}(X),\widehat{S}(Y)\right) = \max\Big\{\sup_{x_t^i\in\widehat{S}(X)}\ \inf_{y_t^i\in\widehat{S}(Y)}\left\|x_t^i - y_t^i\right\|^p,\ \sup_{y_t^i\in\widehat{S}(Y)}\ \inf_{x_t^i\in\widehat{S}(X)}\left\|x_t^i - y_t^i\right\|^p\Big\}\,.$$

(ii) The distance between $\widehat{S}(X)$ and θ^n is denoted by

$$d_H^p\left(\widehat{S}(X),\theta^n\right) = \left\|\widehat{S}(X)\right\|^p = \sup_i \left\|x_t^i\right\|^p = \phi(t)\,.$$

Definition 3.20 *The trivial stochastic set solution of* (3.1) *is said to be*

(p-LSS) *p-Lyapunov stable by selector, if for each $\varepsilon > 0$, there exist $t_0 > 0$ and $\eta = \eta(t_0,\varepsilon)$, such that $\left\|\widehat{S}_0(X)\right\|^p \leq \eta$ implies $\forall T \geq t_0,\ \left\|\widehat{S}(X)\right\|^p < \varepsilon$.*

(p-ALSS) *p-Asymptotical Lyapunov stable for selector if it is (p-LSS) and $\lim\limits_{t\to+\infty}\left\|\widehat{S}(X)\right\|^p = 0\,.$*

(p-ELSS) *p-Exponent Lyapunov stable for selector if there exist $\alpha, \beta > 0$, such that:*

$$\left\|\widehat{S}(X)\right\|^p \leq \beta \cdot \left\|\widehat{S}_0(X)\right\|^p \cdot \exp\left[-\alpha\left(T - t_0\right)\right].$$

Theorem 3.21 *According to Definitions 3.11 and 3.20, we can now say that:*

(i) (p-LSS) $\Rightarrow$ (p-LS).

(ii) (p-LASS) $\Rightarrow$ (p-LAS).

(iii) (p-ELSS) $\Rightarrow$ (p-ELS).

Proof. (i) $E_T\left(\|X_t\|^p\right) = \int\limits_0^T \sup\left\{\left\|x_t^i\right\|^p | x_t^i \in \mathbb{R}^n,\, i \in I \subset \mathbb{N}\right\}\mathrm{d}t \leqslant \sup\left\{\int\limits_0^T \left\|\widehat{S}(X)\right\|^p \mathrm{d}t\right\} \leqslant$ ε_{mp}, where we choose: $\varepsilon_{mp} = T \cdot \max\left\{\varepsilon_i \mid i \in I \subset \mathbb{N}\right\}$.

(ii) $\lim\limits_{T\to\infty} E_T\left(\|X_t\|^p\right) = \lim\limits_{T\to\infty}\left(\sup\left\{\int\limits_0^T \sup\left\|x_t^i\right\|^p \mathrm{d}t\right\}\right) \to 0.$

(iii) Suppose that $T_0 > t$, then from $E_{T_0}\left(\left\|\widehat{S}^p(X)\right\|\right) \leqslant E_{T_0}\left(\|X_t\|^p\right)$. Hence, affirmation (iii) is clear.

□

Theorem 3.22 *Suppose that the positive Lyapunov-like function $V\left(t, x_t^i\right) \in C^{1,2}\left(J \times \mathbb{R}^n, \mathbb{R}^+\right)$ satisfies the following conditions:*

(i) $|V(t, \overline{x_t^i}) - V(t, x_t^i)| \leq L\left\|\overline{x_t^i} - x_t^i\right\|^p$, where L is bounded Lipschitz constant, for all $\overline{x_t^i}, x_t^i \in \mathbb{R}^n$ and $t \in J$;

(ii) the Dini derivative $D^+V(t, X_t) \leqslant g(t, V(t, x_t^i))$.

If x_t^i is selector stochastic solution $x_t^i \in \mathbb{R}^n$ of SSDE (3.1) *with selector and $V(t_0, x_0) \leqslant k_0$, then $V(t, x_t^i) \leqslant l\left(t, t_0, l(t_0)\right)$, where $l\left(t, t_0, l(t_0)\right)$ is a maximal solution of the ordinary differential equation (ODE)*

$$\frac{\mathrm{d}l}{\mathrm{d}t} = g\left(t, l(t)\right), \quad l\left(t_0\right) = k_0 \geqslant 0.$$

Proof. Let $x_t^i = x(t, \omega, t_0, x_0)$ is any selector-stochastic solution $x_t^i \in E^{nN}$ of SSDE (3.1) with selectors existing on J. Define the function $m(t) = V(t, x_t^i)$ so that $m(t_0) = V(t_0, x_0) \leqslant k_0$. Now for small $h > 0$, by our assumption follows that

$$m(t+h) - m(t) = V(t+h, x_{t+h}^i) - V(t, x_t^i)$$

then we have $D^+m(t) = D^+V(t, x_t^i) \leqslant g(t, m(t))$, where $\forall t \geqslant t_0 :\ m(t_0) \leqslant k_0$, and the estimate $\forall t \geqslant t_0 :\ m(t) \leqslant l(t, t_0, k_0)$. □

Corollary 3.23 *If the Lyapunov-like function $V(t, x_t^i)$ satisfies conditions in Theorem 3.22, then we have the estimate:*

$$\forall t \geqslant t_0 > 0 : \ V\left(t, x_t^i\right) \leqslant V\left(t_0, x_0^i\right).$$

Theorem 3.24 *Assume that the positive Lyapunov function $V(t, x_t^i) \in C^{1,2}\left([0, T] \times \mathbb{R}^n, \mathbb{R}^+\right)$ satisfies the following conditions:*

(i) $\left|V(t, x_t^i) - V(t, \overline{x_t^i})\right| < L \cdot (\|x_t^i - \overline{x_t^i}\|^p)$

(ii) There exist $a, b \in \mathbb{R}^+$ such that $b \cdot \phi(t) \leq V(t, x_t^i) \leq a \cdot \phi(t)$

(iii) The Dini derivative $D^+V(t, x_t^i) \leq g(t, V(t, x_t^i))$, where $g(t, 0) = 0$, $g \in C(\mathbb{R}_+^2, \mathbb{R})$.

Then:

(a) If $g(t, V(t, x_t^i)) \leq 0$, then the trivial stochastic set solution is (p-LSS).

(b) If $g(t, V(t, x_t^i)) < 0$, then the trivial stochastic set solution is (p-ALSS).

(c) If $g(t, V(t, x_t^i)) \le -\alpha\, V(t, x_t^i)$, then the trivial stochastic set solution is (p-ELSS).

Proof. (a) We have to prove that:

$$\forall \varepsilon > 0, \exists \eta(t_0, \varepsilon) : \left\| \widehat{S}_0(X) \right\|^p \le \eta \ \text{ implies } \ \forall t \ge t_0 : \ \left\| \widehat{S}(X) \right\|^p < \varepsilon.$$

Because $D^+V\left(t, x_t^i\right) \le g(t, V(t, x_t^i)) \le 0$, then $\forall t \ge t_0,\ V(t, x_t^i) \le V(t_0, x_{t_0}^i)$.
By (ii), this implies $b \cdot \phi(t) \le V(t, x_t^i) \le V(t_0, x_{t_0}^i) \le a \cdot \phi(t_0)$.
Choosing $\eta = \dfrac{b}{a}\varepsilon$, we have $\left\| \widehat{S}(X) \right\|^p < \varepsilon$.

(b) Suppose that $g\left(t, V(t, x_t^i)\right) < 0$. Then the trivial stochastic set solution is (p-LSS). We have to prove that $\lim\limits_{t \to +\infty} \left\| \widehat{S}(X) \right\|^p = 0$. If it is not true, there is ε_0 such that $\left\| \widehat{S}(X) \right\|^p \ge \varepsilon_0$ and $\eta < \varepsilon_0$. On the other hand, we have

$$b\,\varepsilon_0 \leqslant b\,\phi(t) \le V(t, x_t^i) \le V(t_0, x_{t_0}^i) \le a\,\phi(t_0) = a\,\eta < b\,\varepsilon_0$$

This contradiction proves (p-ALSS).

(c) Assume that $g(t, V(t, x_t^i)) \le -\alpha V(t, x_t^i)$, that means $D^+V \le -\alpha V(t, x_t^i)$ and implies

$$\forall t > t_0 :\ V(t, x_t^i) \le V(t_0, x_{t_0}^i) \cdot \exp\left[-\alpha\,(t - t_0)\right].$$

Hence, we infer

$b\,\phi(t) \le V(t, x_t^i) \le V(t_0, x_{t_0}^i) \cdot \exp\left[-\alpha\,(t - t_0)\right] \le a \cdot \phi(t_0) \cdot \exp\left[-\alpha\,(t - t_0)\right]$,
or $\left\| \widehat{S}(X) \right\|^p \le \beta \cdot \left\| \widehat{S}_0(X) \right\|^p \cdot \exp\left[-\alpha\,(t - t_0)\right]$, where $\beta = \dfrac{a}{b}$. We have (p-ELSS).

□

Definition 3.25 *The trivial stochastic set solution of SSDE* (3.1) *is said to be:*

(p-SS1) *p-equi-stable by selector, if for each $\varepsilon > 0$, and $t_0 > 0$ there exists a $\eta = \eta\,(t_0, \varepsilon)$ such that $\left\| \widehat{S}_0(X) \right\|^p < \eta$ implies that $\left\| \widehat{S}(X) \right\|^p < \varepsilon$*

(p-SS2) *p-uniformly stable by selector, if η in (p-SS1) is independent of t_0.*

(p-SS3) *p-quasi-equi-asymptotically stable, if for each $\varepsilon > 0, t_0 > 0$, there exists a $T' = T'\,(t_0, \varepsilon)$ and $\eta_0 = \eta(t_0)$ such that, for $t > t_0 + T'$ and*

$$\left\| \widehat{S}_0(X) \right\|^p < \eta_0 \quad \textit{implies that} \quad \left\| \widehat{S}(X) \right\|^p < \varepsilon\,.$$

(p-SS4) *p-quasi-uniformly-asymptotically stable by selector, if η_0 and T' in (p-SS3) are independent of t_0.*

(p-SS5) *p-equi-asymptotically stable by selector, if (p-SS1) and (p-SS3) hold simultaneously.*

(p-SS6) *p-uniformly asymptotically stable by selector, if (p-SS2) and (p-SS4) hold simultaneously.*

Remark 3.26 *According to Definitions 3.20 and 3.25, we can now say that:*

(i) If a stochastic set solution of SSCDEs with selectors (3.1) is (p-SS1), then it is (p-LSS), that means: (p-SS1) $\Leftrightarrow$ (p-LSS).

(ii) (p-SS6) $\Leftrightarrow$ (p-ALSS).

(iii) (p-SS6) or (p-ALS) $\Rightarrow$ (p-SS3).

(iv) (p-SS6) $\Rightarrow$ (p-SS4).

Putting $S^{\rho}(x_0^i) = \left\{ x_t^i \in \widehat{S}(X) | \left\| \widehat{S}(X) - \widehat{S}_0(X) \right\|^p \leq \rho, \ i \in I \subset \mathbb{N} \right\}$, we have:

Theorem 3.27 *Assume that for SSDE (3.1) with selectors, there exists the Lyapunov-like function $V(t, x_t^i)$ which satisfies the conditions of theorem 3.12.*

a) If there exist positive functions $a(.)$, $b(.)$, strictly increasing such that:

(i) $\forall t \in J \subset \mathbb{R}^+, x_t^i \in \mathbb{R}^n : b(\phi(t))) \leq V(t, x_t^i) \leq a(t, \phi(t)))$ and $g(t, V(t, x_t^i) \leq 0$, then (p-SS1) holds.

Then:

(ii) If $g(t, V(t, x_t^i)) \leq -\mu_1$, then (p-SS3) holds.

(iii) If $g(t, V(t, x_t^i)) < -\mu_1$, then (p-SS5) holds.

b) If there exist positive functions $a(.)$, $b(.)$, strictly increasing such that:

(i) $b(|\phi(t)|)) \leq V(t, x_t^i) \leq a(t, \phi(t))), \forall t \in [0; T] \subset \mathbb{R}^+, x_t^i \in S^{\rho}(x_0^i)$ and $g(t, V(t, x_t^i) \leq 0$, then (p-SS2) holds.

Furthermore, there exists $\eta' > 0$ such that

(ii) If $g(t, V(t, x_t^i)) \leq -\eta' V(t, x_t^i)$, then (p-SS4) holds.

(iii) If $g(t, V(t, x_t^i)) < -\eta' V(t, x_t^i)$, then (p-SS6) holds.

Proof. Let $\alpha > 0$ and t_0 be given, choosing $\beta = \beta(t_0, \alpha)$ such that $a(t_0, \alpha) < b(\beta)$ with this we have (p-SS1). If this is not true, there would exists a the selector-stochastic solution $x_t^i \in \mathbb{R}^n$ of SSDE (3.1) with selectors and $t > t_0$ such that:

$$\left\| \widehat{S}_0(X) \right\|^p = \alpha \ \text{ and } \ \left\| \widehat{S}(X) \right\|^p > \beta \ \text{ with } \ \alpha < \beta.$$

Assumption (a)(i) shows that $V(t, x(t)) \leq V(t_0, x_0), \forall t \geq t_0 \geq 0$ and condition $a(t_0, \alpha) < b(\beta)$ as result, yield:

$$b(\beta) < b(\phi(t))) \leq V(t, x_t^i) \leq V(t_0, x_0^i) \leq a(t_0, \phi(t_0)) \leq a(t_0, \alpha) < b(\beta)$$

This contradiction proves that (p-SS1) holds.

Next, we have to prove that for all $\alpha > 0$, $t_0 \in J$, there exists a $B > 0$ and number $T' = T'(t_0, \alpha) > 0$ such that, $t > t_0 + T'$ and $\left\|\widehat{S}_0(X)\right\|^p < B$ implies that $\left\|\widehat{S}(X)\right\|^p < \alpha$.

Let $\alpha > 0$ and $t_0 > 0$. Choosing $B = B(t_0, \alpha)$ such that $a(t_0, \alpha) < b(B)$ we have (p-SS3). There would exist a selector-stochastic solution $x_t^i \in \mathbb{R}^n$ of SSDE (3.1) with selectors and $t \geqslant t_0 + T' > t_0 \geqslant 0$ such that $|\phi(t_0 + T')| = \alpha$ and $\phi(t) > B$ with $\alpha < \beta$.

Assumption (a-ii) of theorem 3.27 shows that $\forall t \geqslant t_0 > 0 : V(t, x_t^i) \leq V(t_0, x_0^i)$, and yields:

$$b(B) < b\left(\phi(t)\right) \leqslant V(t, x_t^i) \leqslant V(t_0, x_0^i) - \mu_1 \leqslant a\left(t_0, \phi(t_0)\right) - \mu_1 < a(t_0, \alpha) < b(B).$$

This contradiction proves that (p-SS3) holds.

The affirmation for (p-SS5) is proved analogous to the proof of the affirmations for (p-SS1), (p-SS3).

Next, we have to prove that (p-SS2) holds:

Because $g(t, V\left(t, x_t^i\right)) \leq 0$ implies $V(t, x_t^i) \leq V(t_0, x_0^i)$ and

$$\forall t \geq t_0 : \; b(\phi(t)) \leq V\left(t, x_t^i\right) \leq V(t_0, x_0^i) \leq a(t_0, \phi(t_0)).$$

Thus, for all $x_t^i \in S^\rho(x_0^i)$ and all $t_0 \in J$ the affirmation for (p-SS1) holds, that means the affirmation for (p-SS2) holds.

Next, we have to prove that (p-SS4) holds. According to assumption b) of theorem 3.27

i) $b(\phi(t)) \leq V(t, x_t^i) \leq a(t, \phi(t))$

ii) $D^+V(t, x_t^i) \leq g(t, V(t, x_t^i)) \leq -\eta V(t, x_t^i)$

For all $t_0 \in J$ and $t > t_0$, we have

$$V(t, x_t^i) \leq V(t_0, x_0^i)\exp[-\eta(t - t_0)] \leq a(t_0, \phi(t_0))\exp[-\eta(t - t_0)].$$

As a result,

$$\forall t \geq t_0 \;\; b\left(\phi(t)\right) \leq a(t_0, \phi(t_0)) \cdot \exp[-\eta(t - t_0)]$$

and (p-SS4) holds.

The affirmation for (p-SS6) is proved analogous to the proof of the affirmations for (p-SS2), (p-SS4). □

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