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RIEMANN-LIOUVILLE FRACTIONAL LANDAU TYPE INEQUALITIES

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Abstract. We present uniform Riemann-Liouville left and right fractional Landau inequalities over $\mathbb{R}_+$ and $\mathbb{R}_-$, respectively, of fractional orders $1 < \nu < 2$ and $2 < \nu < 3$, and we estimate lower order fractional derivatives. These inequalities are sharp or nearly sharp with completely determined constants. We give applications when $\nu = 1.5$. We finish with a related new Ostrowski like inequality for $\nu > 0$, $\nu \notin \mathbb{N}$.

Keywords: fractional Landau inequality, Riemann-Liouville left and right fractional derivatives.

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1 Introduction

Landau [6] in 1913 proved if $f \in C^2([0,1])$, $\|f\|_\infty = 1$, $\|f''\|_\infty = 4$, then $\|f'\|_\infty \leq 4$, with 4 the best constant, and the result is not necessarily true for an interval of length < 1. In now days Landau's inequality has taken the form [1]

$$\left\|f'\right\|_{p,I} \leq C_p(I) \left\|f\right\|_{p,I}^{\frac{1}{2}} \left\|f''\right\|_{p,I}^{\frac{1}{2}}, \tag{1.1}$$

where $\|\cdot\|_{p,I}$ is the p-norm on the interval I; $p \in [1, \infty]$, $I = \mathbb{R}_+$ or $I = \mathbb{R}$, and $f : I \to \mathbb{R}$ is twice differentiable with $f, f'' \in L_p(I)$. $C_p(I) > 0$ is independent of f. And the best constants are $C_\infty(\mathbb{R}_+) = 2$ and $C_\infty(\mathbb{R}) = \sqrt{2}$.

Research about Landau type inequalities has expanded to many different directions and it is a very active topic of mathematical activity. Here we are concerned about fractional Landau inequalities of Riemann-Liouville type.

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The main motivation to write this article is [7], and especially the result that follows:

Let α be a complex number with $Re\alpha > 0$, where $Re\alpha$ is the real part of α. We define I_L^α and D_L^α to be the classical Riemann-Liouville fractional integration and differentiation operators of order α,

$$I_L^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, \mathrm{d}t,$$

$$D_L^\alpha f(x) = \frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}} \left(I_L^{n+1-\alpha} f \right)(x),$$

where Γ is the gamma function and $n = [Re\alpha]$ (integral part).

The above operators are defined in the natural domain of appropriate function spaces.

We state

Theorem 1.1 ([7]) *Let I_L^α and D_L^α be the fractional Riemann-Liouville operators in the Fréchet space $L_{loc}^1([0,+\infty))$ with the usual topology. If α, β and γ are complex numbers for which $0 < Re\alpha < Re\beta < Re\gamma$, for every seminorm $\|\cdot\|_p$ there exist constants $F_1(p,\alpha,\beta,\gamma)$ and $F_2(p,\alpha,\beta,\gamma)$ that depend on p, α, β, and γ, with the properties that*

$$\left\| I_L^\beta f \right\|_p \leq F_1(p,\alpha,\beta,\gamma) \left\| I_L^\alpha f \right\|_p^{\frac{Re(\gamma-\beta)}{Re(\gamma-\alpha)}} \left\| I_L^\gamma f \right\|_p^{\frac{Re(\beta-\alpha)}{Re(\gamma-\alpha)}}, \quad f \in L_{loc}^1([0,+\infty)), \tag{1.2}$$

and

$$\left\| D_L^\beta f \right\|_p \leq F_2(p,\alpha,\beta,\gamma) \left\| D_L^\alpha f \right\|_p^{\frac{Re(\gamma-\beta)}{Re(\gamma-\alpha)}} \left\| D_L^\gamma f \right\|_p^{\frac{Re(\beta-\alpha)}{Re(\gamma-\alpha)}}, \tag{1.3}$$

$f \in \mathcal{D}\left(D_L^\gamma\right)$ (domain of D_L^γ).

If $Im\alpha = Im\beta = Im\gamma$, the constants $F_1(p,\alpha,\beta,\gamma)$ and $F_2(p,\alpha,\beta,\gamma)$ depend only on p and on an arbitrary integer $n > Re\gamma$.

For more see [7]. Other inspirations are the papers [3, 4].

The above motivating results and references are existential without giving exact constants which are not the best possible. Their methods are based on functional analysis and semigroup theory, in particular dealing with moment inequalities for fractional powers of operators in Banach spaces.

As a last motivation we also mention [2]. Our method is classical analytic and we will produce fractional Landau type inequalities, involving the Riemann-Liouville left and right fractional derivatives, that are sharp or nearly sharp and all constants are exactly calculated.

We need the following background (based on [8, Section 2] and [5, Chapter 2]).

Let $-\infty < a < b < \infty$, the left and right Riemann-Liouville fractional integrals of order $\alpha \in \mathbb{C}$ $(Re\alpha > 0)$ are defined by

$$\left(I_a^\alpha f \right)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, \mathrm{d}t, \qquad x > a \tag{1.4}$$

and

$$\left(I_{b-}^\alpha f \right)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, \mathrm{d}t, \qquad x < b \tag{1.5}$$

respectively.

The Riemann-Liouville left and right fractional derivatives of order $\alpha \in \mathbb{C}$ ($Re\alpha \geq 0$) are defined by

$$\left(D_a^{\alpha} y\right)(x) := \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(I_a^{n-\alpha} y\right)(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \int_a^x (x-t)^{n-\alpha-1} y(t)\, \mathrm{d}t, \tag{1.6}$$

where $n = [Re\alpha] + 1$; $x > a$, and

$$\left(D_{b-}^{\alpha} y\right)(x) := (-1)^n \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(I_{b-}^{n-\alpha} y\right)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \int_x^b (t-x)^{n-\alpha-1} y(t)\, \mathrm{d}t, \tag{1.7}$$

where $n = [Re\alpha] + 1$; $x < b$, respectively.

In particular, when $\alpha = n \in \mathbb{Z}_+$, then

$$\left(D_a^0 y\right)(x) = \left(D_{b-}^0 y\right)(x) = y(x);$$

$$\left(D_a^n y\right)(x) = y^{(n)}(x), \text{ and } \left(D_{b-}^n y\right)(x) = (-1)^n y^{(n)}(x), \quad n \in \mathbb{N}.$$

We denote by

$$\left(D_a^{-\alpha} f\right)(x) := \left(I_a^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, \mathrm{d}t, \quad x > a, \quad Re\alpha > 0 \tag{1.8}$$

and

$$\left(D_{b-}^{-\alpha} f\right)(x) := \left(I_{b-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, \mathrm{d}t, \quad x < b, \quad Re\alpha > 0. \tag{1.9}$$

Based on [8, p. 44-45] and [5, p. 74-76], we will use the following two fractional Taylor theorems. The left one first:

Theorem 1.2 *Let $\nu > 0$, $\nu \notin \mathbb{N}$ and $m = \lceil \nu \rceil$ (ceiling of ν); φ, $D_a^{\nu}\varphi \in C([a,b])$. Then*

$$\varphi(b) = \sum_{j=1}^{m} \frac{D_a^{\nu-j}\varphi(a)}{\Gamma(\nu-j+1)} (b-a)^{\nu-j} + \frac{1}{\Gamma(\nu)} \int_a^b (b-t)^{\nu-1} D_a^{\nu}\varphi(t)\, \mathrm{d}t. \tag{1.10}$$

It follows the right one:

Theorem 1.3 *Let $\nu > 0$, $\nu \notin \mathbb{N}$ and $m = \lceil \nu \rceil$; φ, $D_{b-}^{\nu}\varphi \in C([a,b])$. Then*

$$\varphi(a) = \sum_{j=1}^{m} \frac{D_{b-}^{\nu-j}\varphi(b)}{\Gamma(\nu-j+1)} (b-a)^{\nu-j} + \frac{1}{\Gamma(\nu)} \int_a^b (t-a)^{\nu-1} D_{b-}^{\nu}\varphi(t)\, \mathrm{d}t. \tag{1.11}$$

2 Main Results

First we give left Riemann-Liouville fractional Landau type inequalities.

Convention 2.1 *Define $D_a^{\nu}\varphi(x) := 0$, for any $x < a$, for any $\nu > 0$, $\nu \notin \mathbb{N}$.*

We present

Theorem 2.2 *Here $1 < \nu < 2$ and $\varphi \in C_B(\mathbb{R}_+)$ and $D_a^{\nu}\varphi \in C_B([a, +\infty))$, $\forall\, a \in \mathbb{R}_+$, where C_B means continuous and bounded functions. We assume that $\sup_{a\in\mathbb{R}_+} \|D_a^{\nu}\varphi\|_{\infty,\mathbb{R}_+} < +\infty$. Then*

$$\sup_{a\in\mathbb{R}_+} \left|D_a^{\nu-1}\varphi(a)\right| \leq \left(\frac{\Gamma(\nu+1)}{(\nu-1)^{\nu-1}}\right)^{\frac{1}{\nu}} \|\varphi\|_{\infty,\mathbb{R}_+}^{\frac{1}{\nu}} \left(\sup_{a\in\mathbb{R}_+} \|D_a^{\nu}\varphi\|_{\infty,\mathbb{R}_+}\right)^{\frac{\nu-1}{\nu}}, \tag{2.1}$$

a sharp inequality with a precisely determined constant. That is $\sup_{a\in\mathbb{R}_+} \left|D_a^{\nu-1}\varphi(a)\right| < +\infty$.

Proof. For any $b \in (a, +\infty)$ we get (by Theorem 1.2)

$$\varphi(b) = \frac{D_a^{\nu-1}\varphi(a)}{\Gamma(\nu)}(b-a)^{\nu-1} + \frac{D_a^{\nu-2}\varphi(a)}{\Gamma(\nu-1)}(b-a)^{\nu-2} + \frac{1}{\Gamma(\nu)}\int_a^b (b-t)^{\nu-1} D_a^{\nu}\varphi(t)\,dt. \tag{2.2}$$

Notice that ($x \in [a, +\infty)$)

$$D_a^{\nu-2}\varphi(x) = D_a^{-(2-\nu)}\varphi(x) = \frac{1}{\Gamma(2-\nu)}\int_a^x (x-t)^{1-\nu}\varphi(t)\,dt,$$

and

$$\begin{aligned}\left|D_a^{\nu-2}\varphi(x)\right| &\leq \frac{1}{\Gamma(2-\nu)}\int_a^x (x-t)^{1-\nu}|\varphi(t)|\,dt \leq \frac{\|\varphi\|_{\infty,[a,+\infty)}}{\Gamma(2-\nu)}\int_a^x (x-t)^{1-\nu}\,dt \\ &= \frac{\|\varphi\|_{\infty,[a,+\infty)}}{\Gamma(2-\nu)} \cdot \frac{(x-a)^{2-\nu}}{(2-\nu)} = \frac{\|\varphi\|_{\infty,[a,+\infty)}}{\Gamma(3-\nu)}(x-a)^{2-\nu}.\end{aligned} \tag{2.3}$$

That is

$$\left|D_a^{\nu-2}\varphi(x)\right| \leq \frac{\|\varphi\|_{\infty,[a,+\infty)}}{\Gamma(3-\nu)}(x-a)^{2-\nu}, \quad \forall x \in [a, +\infty). \tag{2.4}$$

Clearly, it is $D_a^{\nu-2}\varphi(a) = 0$.

Hence by (2.2) we get

$$\varphi(b) = \frac{D_a^{\nu-1}\varphi(a)(b-a)^{\nu-1}}{\Gamma(\nu)} + \frac{1}{\Gamma(\nu)}\int_a^b (b-t)^{\nu-1} D_a^{\nu}\varphi(t)\,dt, \quad \forall b \in (a, +\infty). \tag{2.5}$$

So that

$$\frac{D_a^{\nu-1}\varphi(a)}{\Gamma(\nu)}(b-a)^{\nu-1} = \varphi(b) - \frac{1}{\Gamma(\nu)}\int_a^b (b-t)^{\nu-1} D_a^{\nu}\varphi(t)\,dt, \quad \forall b \in (a, +\infty). \tag{2.6}$$

Hence (set $h := b - a > 0$)

$$\begin{aligned}\left|D_a^{\nu-1}\varphi(a)\right| h^{\nu-1} &= \left|\Gamma(\nu)\varphi(b) - \int_a^b (b-t)^{\nu-1} D_a^{\nu}\varphi(t)\,\mathrm{d}t\right| \\ &\le \Gamma(\nu)|\varphi(b)| + \int_a^b (b-t)^{\nu-1}\left|D_a^{\nu}\varphi(t)\right|\mathrm{d}t \\ &\le \Gamma(\nu)|\varphi(b)| + \|D_a^{\nu}\varphi\|_{\infty,[a,+\infty)}\int_a^b (b-t)^{\nu-1}\,\mathrm{d}t \\ &= \Gamma(\nu)|\varphi(b)| + \|D_a^{\nu}\varphi\|_{\infty,[a,+\infty)}\frac{(b-a)^{\nu}}{\nu} \\ &= \Gamma(\nu)|\varphi(b)| + \|D_a^{\nu}\varphi\|_{\infty,[a,+\infty)}\frac{h^{\nu}}{\nu}. \end{aligned} \tag{2.7}$$

That is we have

$$\begin{aligned}\left|D_a^{\nu-1}\varphi(a)\right| h^{\nu-1} &\le \Gamma(\nu)|\varphi(b)| + \|D_a^{\nu}\varphi\|_{\infty,[a,+\infty)}\frac{h^{\nu}}{\nu} \\ &\le \Gamma(\nu)\|\varphi\|_{\infty,\mathbb{R}_+} + \frac{h^{\nu}}{\nu}\cdot\sup_{a\in\mathbb{R}_+}\|D_a^{\nu}\varphi\|_{\infty,[a,+\infty)}. \end{aligned} \tag{2.8}$$

Furthermore we see that

$$\left|D_a^{\nu-1}\varphi(a)\right| \le \frac{\Gamma(\nu)\|\varphi\|_{\infty,\mathbb{R}_+}}{h^{\nu-1}} + \frac{h}{\nu}\cdot\sup_{a\in\mathbb{R}_+}\|D_a^{\nu}\varphi\|_{\infty,[a,+\infty)}, \quad \forall h>0 \text{ and } \forall a\in\mathbb{R}_+. \tag{2.9}$$

Consequently it holds

$$\sup_{a\in\mathbb{R}_+}\left|D_a^{\nu-1}\varphi(a)\right| \le \frac{\Gamma(\nu)\|\varphi\|_{\infty,\mathbb{R}_+}}{h^{\nu-1}} + \frac{h}{\nu}\cdot\sup_{a\in\mathbb{R}_+}\|D_a^{\nu}\varphi\|_{\infty,\mathbb{R}_+}, \quad \forall h>0. \tag{2.10}$$

Call

$$\mu := \Gamma(\nu)\|\varphi\|_{\infty,\mathbb{R}_+}, \tag{2.11}$$

$$\theta := \frac{\sup_{a\in\mathbb{R}_+}\|D_a^{\nu}\varphi\|_{\infty,\mathbb{R}_+}}{\nu}, \tag{2.12}$$

both are greater than zero.

Here $1<\nu<2$ and $0<\nu-1<1$. Set also $\rho := \nu-1$, *i.e.* $0<\rho<1$.

We consider the function

$$y(h) := \mu h^{-\rho} + \theta h, \ \ \forall\, h>0. \tag{2.13}$$

We have

$$y'(h) = -\rho\mu h^{-(\rho+1)} + \theta = 0 \tag{2.14}$$

and

$$\rho\mu h^{-(\rho+1)} = \theta,$$

then

$$h^{-(\rho+1)} = \frac{\theta}{\rho\mu},$$

with the only critical number

$$h_0 = \left(\frac{\rho\mu}{\theta}\right)^{\frac{1}{\rho+1}} > 0. \tag{2.15}$$

Also it holds

$$y''(h_0) = \rho(\rho+1)\mu h_0^{-\rho-2} > 0. \tag{2.16}$$

Thus y has a global minimum over $\mathbb{R}_+$ which is

$$y(h_0) = \mu h_0^{-\rho} + \theta h_0 = \frac{(\rho+1)}{\rho^{\frac{\rho}{\rho+1}}}\theta^{\frac{\rho}{\rho+1}}\mu^{\frac{1}{\rho+1}}. \tag{2.17}$$

Consequently we derive that

$$\sup_{a\in\mathbb{R}_+}\left|D_a^{\nu-1}\varphi(a)\right| \leq \left(\frac{\nu}{(\nu-1)^{\frac{\nu-1}{\nu}}}\right)\left(\frac{\sup_{a\in\mathbb{R}_+}\|D_a^{\nu}\varphi\|_{\infty,\mathbb{R}_+}}{\nu}\right)^{\frac{\nu-1}{\nu}}\left(\Gamma(\nu)\|\varphi\|_{\infty,\mathbb{R}_+}\right)^{\frac{1}{\nu}}$$

$$= \left(\frac{\Gamma(\nu+1)}{(\nu-1)^{\nu-1}}\right)^{\frac{1}{\nu}}\|\varphi\|_{\infty,\mathbb{R}_+}^{\frac{1}{\nu}}\left(\sup_{a\in\mathbb{R}_+}\|D_a^{\nu}\varphi\|_{\infty,\mathbb{R}_+}\right)^{\frac{\nu-1}{\nu}}, \text{ for } 1<\nu<2. \tag{2.18}$$

The theorem is proved. □

We continue with

Theorem 2.3 *Here* $2<\nu<3$ *and* $\varphi \in C_B(\mathbb{R}_+)$ *and* $D_a^{\nu}\varphi \in C_B([a,+\infty))$, $\forall\, a\in\mathbb{R}_+$. *We assume that* $\sup_{a\in\mathbb{R}_+}\|D_a^{\nu}\varphi\|_{\infty,\mathbb{R}_+} < +\infty$. *Then*

i)

$$\sup_{a\in\mathbb{R}_+}\left|D_a^{\nu-1}\varphi(a)\right| \leq \left(\frac{5}{\nu-1}\right)^{\frac{\nu-1}{\nu}}\left(\nu\left(1+2^{2-\nu}\right)\Gamma(\nu)\right)^{\frac{1}{\nu}}\|\varphi\|_{\infty,\mathbb{R}_+}^{\frac{1}{\nu}}\left(\sup_{a\in\mathbb{R}_+}\|D_a^{\nu}\varphi\|_{\infty,\mathbb{R}_+}\right)^{\frac{\nu-1}{\nu}}, \tag{2.19}$$

ii)

$$\sup_{a\in\mathbb{R}_+}\left|D_a^{\nu-2}\varphi(a)\right| \leq \left(\frac{6}{(\nu-1)(\nu-2)}\right)^{\frac{\nu-2}{\nu}}\left(\nu\left(1+2^{1-\nu}\right)\Gamma(\nu-1)\right)^{\frac{2}{\nu}}\|\varphi\|_{\infty,\mathbb{R}_+}^{\frac{2}{\nu}}\left(\sup_{a\in\mathbb{R}_+}\|D_a^{\nu}\varphi\|_{\infty,\mathbb{R}_+}\right)^{\frac{\nu-2}{\nu}}. \tag{2.20}$$

The above inequalities are nearly sharp with precisely determined constants.

That is $\sup_{a\in\mathbb{R}_+}\left|D_a^{\nu-1}\varphi(a)\right|, \sup_{a\in\mathbb{R}_+}\left|D_a^{\nu-2}\varphi(a)\right| < +\infty$.

Proof. For any $b \in (a, +\infty)$ we get (by Theorem 1.2)

$$\begin{aligned} \varphi(b) &= \frac{D_a^{\nu-1}\varphi(a)}{\Gamma(\nu)}(b-a)^{\nu-1} + \frac{D_a^{\nu-2}\varphi(a)}{\Gamma(\nu-1)}(b-a)^{\nu-2} \\ &+ \frac{D_a^{\nu-3}\varphi(a)}{\Gamma(\nu-2)}(b-a)^{\nu-3} + \frac{1}{\Gamma(\nu)}\int_a^b (b-t)^{\nu-1} D_a^{\nu}\varphi(t)\,\mathrm{d}t. \end{aligned} \tag{2.21}$$

Notice that ($x \in [a, +\infty)$)

$$D_a^{\nu-3}\varphi(x) = D_a^{-(3-\nu)}\varphi(x) = \frac{1}{\Gamma(3-\nu)}\int_a^x (x-t)^{2-\nu}\varphi(t)\,\mathrm{d}t,$$

and

$$\begin{aligned} \left|D_a^{\nu-3}\varphi(x)\right| &\le \frac{1}{\Gamma(3-\nu)}\int_a^x (x-t)^{2-\nu}|\varphi(t)|\,\mathrm{d}t \le \frac{\|\varphi\|_{\infty,[a,+\infty)}}{\Gamma(3-\nu)}\int_a^x (x-t)^{2-\nu}\,\mathrm{d}t \\ &= \frac{\|\varphi\|_{\infty,[a,+\infty)}}{\Gamma(3-\nu)} \cdot \frac{(x-a)^{3-\nu}}{(3-\nu)} = \frac{\|\varphi\|_{\infty,[a,+\infty)}}{\Gamma(4-\nu)}(x-a)^{3-\nu}. \end{aligned} \tag{2.22}$$

That is

$$\left|D_a^{\nu-3}\varphi(x)\right| \le \frac{\|\varphi\|_{\infty,[a,+\infty)}}{\Gamma(4-\nu)}(x-a)^{3-\nu}, \quad \forall x \in [a, +\infty). \tag{2.23}$$

Clearly, it is $D_a^{\nu-3}\varphi(a) = 0$.

Hence by (2.21), for all $b \in (a, +\infty)$, we get

$$\varphi(b) = \frac{D_a^{\nu-1}\varphi(a)}{\Gamma(\nu)}(b-a)^{\nu-1} + \frac{D_a^{\nu-2}\varphi(a)}{\Gamma(\nu-1)}(b-a)^{\nu-2} + \frac{1}{\Gamma(\nu)}\int_a^b (b-t)^{\nu-1} D_a^{\nu}\varphi(t)\,\mathrm{d}t. \tag{2.24}$$

So that, for all $b \in (a, +\infty)$

$$\frac{D_a^{\nu-1}\varphi(a)}{\Gamma(\nu)}(b-a)^{\nu-1} + \frac{D_a^{\nu-2}\varphi(a)}{\Gamma(\nu-1)}(b-a)^{\nu-2} = \varphi(b) - \frac{1}{\Gamma(\nu)}\int_a^b (b-t)^{\nu-1} D_a^{\nu}\varphi(t)\,\mathrm{d}t. \tag{2.25}$$

Let $b = a + h$, $h > 0$, *i.e.* $b - a = h$, then

$$\begin{aligned} &\frac{D_a^{\nu-1}\varphi(a)}{\Gamma(\nu)}h^{\nu-1} + \frac{D_a^{\nu-2}\varphi(a)}{\Gamma(\nu-1)}h^{\nu-2} \\ &= \varphi(a+h) - \frac{1}{\Gamma(\nu)}\int_a^{a+h} (a+h-t)^{\nu-1} D_a^{\nu}\varphi(t)\,\mathrm{d}t =: A. \end{aligned} \tag{2.26}$$

For $b = a + 2h$, *i.e.* $b - a = 2h$, we get

$$\begin{aligned} &\frac{D_a^{\nu-1}\varphi(a)}{\Gamma(\nu)}2^{\nu-1}h^{\nu-1} + \frac{D_a^{\nu-2}\varphi(a)}{\Gamma(\nu-1)}2^{\nu-2}h^{\nu-2} \\ &= \varphi(a+2h) - \frac{1}{\Gamma(\nu)}\int_a^{a+2h} (a+2h-t)^{\nu-1} D_a^{\nu}\varphi(t)\,\mathrm{d}t =: B. \end{aligned} \tag{2.27}$$

We solve the system of two equations (2.26), (2.27) for two unknowns $D_a^{\nu-1}\varphi(a)$ and $D_a^{\nu-2}\varphi(a)$. We calculate the determinants:

$$D=\begin{vmatrix}\frac{h^{\nu-1}}{\Gamma(\nu)} & \frac{h^{\nu-2}}{\Gamma(\nu-1)}\\ \frac{2^{\nu-1}h^{\nu-1}}{\Gamma(\nu)} & \frac{2^{\nu-2}h^{\nu-2}}{\Gamma(\nu-1)}\end{vmatrix}=\frac{2^{\nu-2}h^{2\nu-3}-2^{\nu-1}h^{2\nu-3}}{\Gamma(\nu)\Gamma(\nu-1)}$$

$$=\frac{h^{2\nu-3}}{\Gamma(\nu)\Gamma(\nu-1)}\left(2^{\nu-2}-2^{\nu-1}\right)=-\frac{2^{\nu-2}h^{2\nu-3}}{\Gamma(\nu)\Gamma(\nu-1)}. \tag{2.28}$$

Hence we get

$$D=-\frac{2^{\nu-2}h^{2\nu-3}}{\Gamma(\nu)\Gamma(\nu-1)}<0. \tag{2.29}$$

Next we find

$$D_1=\begin{vmatrix}A & \frac{h^{\nu-2}}{\Gamma(\nu-1)}\\ B & \frac{2^{\nu-2}h^{\nu-2}}{\Gamma(\nu-1)}\end{vmatrix}=\frac{h^{\nu-2}}{\Gamma(\nu-1)}\left(2^{\nu-2}A-B\right), \tag{2.30}$$

and

$$D_2=\begin{vmatrix}\frac{h^{\nu-1}}{\Gamma(\nu)} & A\\ \frac{2^{\nu-1}h^{\nu-1}}{\Gamma(\nu)} & B\end{vmatrix}=\frac{h^{\nu-1}}{\Gamma(\nu)}\left(B-2^{\nu-1}A\right). \tag{2.31}$$

Therefore we derive

$$D_a^{\nu-1}\varphi(a)=\frac{D_1}{D}=\frac{\frac{h^{\nu-2}}{\Gamma(\nu-1)}\left(2^{\nu-2}A-B\right)}{-\frac{2^{\nu-2}h^{2\nu-3}}{\Gamma(\nu)\Gamma(\nu-1)}}=\frac{\Gamma(\nu)\left(B-2^{\nu-2}A\right)}{2^{\nu-2}h^{\nu-1}}. \tag{2.32}$$

Similarly, we get

$$D_a^{\nu-2}\varphi(a)=\frac{D_2}{D}=\frac{\frac{h^{\nu-1}}{\Gamma(\nu)}\left(B-2^{\nu-1}A\right)}{-\frac{2^{\nu-2}h^{2\nu-3}}{\Gamma(\nu)\Gamma(\nu-1)}}=\frac{\Gamma(\nu-1)\left(2^{\nu-1}A-B\right)}{2^{\nu-2}h^{\nu-2}}. \tag{2.33}$$

We notice that

$$|A|=\left|\varphi(a+h)-\frac{1}{\Gamma(\nu)}\int_a^{a+h}(a+h-t)^{\nu-1}D_a^{\nu}\varphi(t)\,\mathrm{d}t\right|$$

$$\leq\|\varphi\|_{\infty,\mathbb{R}_+}+\frac{h^{\nu}}{\Gamma(\nu+1)}\sup_{a\in\mathbb{R}_+}\|D_a^{\nu}\varphi\|_{\infty,\mathbb{R}_+}, \tag{2.34}$$

and

$$|B|=\left|\varphi(a+2h)-\frac{1}{\Gamma(\nu)}\int_a^{a+2h}(a+2h-t)^{\nu-1}D_a^{\nu}\varphi(t)\,\mathrm{d}t\right|$$

$$\leq\|\varphi\|_{\infty,\mathbb{R}_+}+\frac{2^{\nu}h^{\nu}}{\Gamma(\nu+1)}\sup_{a\in\mathbb{R}_+}\|D_a^{\nu}\varphi\|_{\infty,\mathbb{R}_+}. \tag{2.35}$$

Therefore we have

$$\left|D_a^{\nu-1}\varphi(a)\right|\leq\frac{\Gamma(\nu)\left(|B|+2^{\nu-2}|A|\right)}{2^{\nu-2}h^{\nu-1}}\leq\frac{\Gamma(\nu)}{2^{\nu-2}h^{\nu-1}}.$$

$$\left(\|\varphi\|_{\infty,\mathbb{R}_+}+\frac{2^{\nu}h^{\nu}}{\Gamma(\nu+1)}\sup_{a\in\mathbb{R}_+}\|D_a^{\nu}\varphi\|_{\infty,\mathbb{R}_+}+2^{\nu-2}\|\varphi\|_{\infty,\mathbb{R}_+}+\frac{2^{\nu-2}h^{\nu}}{\Gamma(\nu+1)}\sup_{a\in\mathbb{R}_+}\|D_a^{\nu}\varphi\|_{\infty,\mathbb{R}_+}\right)$$
$$=\frac{\Gamma(\nu)}{2^{\nu-2}h^{\nu-1}}\left[\left(2^{\nu-2}+1\right)\|\varphi\|_{\infty,\mathbb{R}_+}+\frac{\left(2^{\nu}+2^{\nu-2}\right)h^{\nu}}{\Gamma(\nu+1)}\sup_{a\in\mathbb{R}_+}\|D_a^{\nu}\varphi\|_{\infty,\mathbb{R}_+}\right]$$
$$=\frac{\Gamma(\nu)}{2^{\nu-2}h^{\nu-1}}\left[\left(2^{\nu-2}+1\right)\|\varphi\|_{\infty,\mathbb{R}_+}+\frac{\left(5\cdot 2^{\nu-2}\right)h^{\nu}}{\Gamma(\nu+1)}\sup_{a\in\mathbb{R}_+}\|D_a^{\nu}\varphi\|_{\infty,\mathbb{R}_+}\right]$$
$$=\frac{\left(1+2^{2-\nu}\right)\Gamma(\nu)\|\varphi\|_{\infty,\mathbb{R}_+}}{h^{\nu-1}}+\frac{5h}{\nu}\sup_{a\in\mathbb{R}_+}\|D_a^{\nu}\varphi\|_{\infty,\mathbb{R}_+}. \tag{2.36}$$

That is, for all $h>0,\ 2<\nu<3$,

$$\sup_{a\in\mathbb{R}_+}\left|D_a^{\nu-1}\varphi(a)\right|\leq\frac{\left(1+2^{2-\nu}\right)\Gamma(\nu)\|\varphi\|_{\infty,\mathbb{R}_+}}{h^{\nu-1}}+\frac{5h}{\nu}\sup_{a\in\mathbb{R}_+}\|D_a^{\nu}\varphi\|_{\infty,\mathbb{R}_+}. \tag{2.37}$$

Furthermore we have

$$\left|D_a^{\nu-2}\varphi(a)\right|\leq\frac{\Gamma(\nu-1)}{2^{\nu-2}h^{\nu-2}}\left(2^{\nu-1}|A|+|B|\right)$$
$$\leq\frac{\Gamma(\nu-1)}{2^{\nu-2}h^{\nu-2}}\left[2^{\nu-1}\left(\|\varphi\|_{\infty,\mathbb{R}_+}+\frac{h^{\nu}}{\Gamma(\nu+1)}\sup_{a\in\mathbb{R}_+}\|D_a^{\nu}\varphi\|_{\infty,\mathbb{R}_+}\right)\right.$$
$$\left.+\left(\|\varphi\|_{\infty,\mathbb{R}_+}+\frac{2^{\nu}h^{\nu}}{\Gamma(\nu+1)}\sup_{a\in\mathbb{R}_+}\|D_a^{\nu}\varphi\|_{\infty,\mathbb{R}_+}\right)\right]$$
$$=\frac{\Gamma(\nu-1)}{2^{\nu-2}h^{\nu-2}}\left[\left(2^{\nu-1}+1\right)\|\varphi\|_{\infty,\mathbb{R}_+}+\frac{\left(2^{\nu-1}+2^{\nu}\right)h^{\nu}}{\Gamma(\nu+1)}\sup_{a\in\mathbb{R}_+}\|D_a^{\nu}\varphi\|_{\infty,\mathbb{R}_+}\right]$$
$$=\left[\frac{\left(2^{\nu-1}+1\right)\Gamma(\nu-1)\|\varphi\|_{\infty,\mathbb{R}_+}}{2^{\nu-2}h^{\nu-2}}+\frac{\left(2^{\nu-1}+2^{\nu}\right)\Gamma(\nu-1)}{2^{\nu-2}\Gamma(\nu+1)h^{2}}\sup_{a\in\mathbb{R}_+}\|D_a^{\nu}\varphi\|_{\infty,\mathbb{R}_+}\right]$$
$$=\frac{2\left(1+2^{1-\nu}\right)\Gamma(\nu-1)\|\varphi\|_{\infty,\mathbb{R}_+}}{h^{\nu-2}}+\frac{6h^2}{\nu(\nu-1)}\sup_{a\in\mathbb{R}_+}\|D_a^{\nu}\varphi\|_{\infty,\mathbb{R}_+}. \tag{2.38}$$

That is, for all $h>0,\ 2<\nu<3$,

$$\sup_{a\in\mathbb{R}_+}\left|D_a^{\nu-2}\varphi(a)\right|\leq\frac{2\left(1+2^{1-\nu}\right)\Gamma(\nu-1)\|\varphi\|_{\infty,\mathbb{R}_+}}{h^{\nu-2}}+\frac{6h^2}{\nu(\nu-1)}\sup_{a\in\mathbb{R}_+}\|D_a^{\nu}\varphi\|_{\infty,\mathbb{R}_+}. \tag{2.39}$$

Next we are working on (2.37).

Call

$$\mu:=\left(1+2^{2-\nu}\right)\Gamma(\nu)\|\varphi\|_{\infty,\mathbb{R}_+},$$
$$\theta:=\tfrac{5}{\nu}\sup_{a\in\mathbb{R}_+}\|D_a^{\nu}\varphi\|_{\infty,\mathbb{R}_+}, \tag{2.40}$$

both are greater than zero.

Here $2<\nu<3$ and $1<\nu-1<2$. Set also $\rho:=\nu-1$, *i.e.* $1<\rho<2$.

We consider again the function

$$y(h) := \mu h^{-\rho} + \theta h, \ \ \forall\, h > 0. \tag{2.41}$$

As in the proof of Theorem 2.2, y has a global minimum at $h_0 = \left(\frac{\rho\mu}{\theta}\right)^{\frac{1}{\rho+1}} > 0$, which is

$$y(h_0) = \frac{\rho+1}{\rho^{\frac{\rho}{\rho+1}}} \theta^{\frac{\rho}{\rho+1}} \mu^{\frac{1}{\rho+1}}. \tag{2.42}$$

Consequently we derive

$$\begin{aligned}
&\sup_{a\in\mathbb{R}_+} \left| D_a^{\nu-1}\varphi(a) \right| \\
&\leq \frac{\nu}{(\nu-1)^{\frac{\nu-1}{\nu}}} \left(\frac{5}{\nu} \sup_{a\in\mathbb{R}_+} \left\| D_a^{\nu}\varphi \right\|_{\infty,\mathbb{R}_+} \right)^{\frac{\nu-1}{\nu}} \left(\left(1+2^{2-\nu}\right) \Gamma(\nu) \left\|\varphi\right\|_{\infty,\mathbb{R}_+} \right)^{\frac{1}{\nu}} \\
&= \left(\frac{5}{\nu-1}\right)^{\frac{\nu-1}{\nu}} \left(\nu\left(1+2^{2-\nu}\right)\Gamma(\nu)\right)^{\frac{1}{\nu}} \left\|\varphi\right\|_{\infty,\mathbb{R}_+}^{\frac{1}{\nu}} \left(\sup_{a\in\mathbb{R}_+} \left\| D_a^{\nu}\varphi \right\|_{\infty,\mathbb{R}_+} \right)^{\frac{\nu-1}{\nu}}, 2<\nu<3.
\end{aligned} \tag{2.43}$$

Next we are working on (2.39).

Call

$$\xi := 2\left(1+2^{1-\nu}\right)\Gamma(\nu-1) \left\|\varphi\right\|_{\infty,\mathbb{R}_+}, \tag{2.44}$$

$$\psi := \frac{6}{\nu(\nu-1)} \sup_{a\in\mathbb{R}_+} \left\| D_a^{\nu}\varphi \right\|_{\infty,\mathbb{R}_+}, \tag{2.45}$$

both are greater than zero.

Call also $\rho := \nu - 2$, $2 < \nu < 3$.

We consider the function

$$y(h) := \xi h^{-\rho} + \psi h^2, \ \ \forall\, h > 0. \tag{2.46}$$

Hence

$$y'(h) = -\rho\xi h^{-(\rho+1)} + 2\psi h = 0,$$

with the only critical number

$$h_0 = h_{crit.} = \left(\frac{\rho\xi}{2\psi}\right)^{\frac{1}{\rho+2}}. \tag{2.47}$$

We need

$$y''(h) = \rho(\rho+1)\xi h^{-(\rho+2)} + 2\psi. \tag{2.48}$$

We calculate

$$\begin{aligned}
y''(h_0) &= \rho(\rho+1)\xi h_0^{-(\rho+2)} + 2\psi = \rho(\rho+1)\xi \left(\left(\frac{\rho\xi}{2\psi}\right)^{\frac{1}{\rho+2}} \right)^{-(\rho+2)} + 2\psi \\
&= \rho(\rho+1)\xi\left(\frac{2\psi}{\rho\xi}\right) + 2\psi = (\rho+1)2\psi + 2\psi = 2\psi(\rho+2) > 0.
\end{aligned} \tag{2.49}$$

Thus y has a global minimum, which is

$$\begin{aligned} y(h_0) &= \xi h_0^{-\rho} + \psi h_0^2 = h_0^2\left(\xi h_0^{-(\rho+2)} + \psi\right) = \left(\frac{\rho\xi}{2\psi}\right)^{\frac{2}{\rho+2}}\left(\xi\left(\left(\frac{\rho\xi}{2\psi}\right)^{\frac{1}{\rho+2}}\right)^{-(\rho+2)} + \psi\right) \\ &= \left(\frac{\rho\xi}{2\psi}\right)^{\frac{2}{\rho+2}}\left(\xi\left(\frac{2\psi}{\rho\xi}\right) + \psi\right) = \left(\frac{\rho\xi}{2\psi}\right)^{\frac{2}{\rho+2}}\left(\frac{2\psi}{\rho} + \psi\right) \\ &= \psi\left(\frac{\rho+2}{\rho}\right)\left(\frac{\rho\xi}{2\psi}\right)^{\frac{2}{\rho+2}} = (\rho+2)\left(\frac{\xi}{2}\right)^{\frac{2}{2+\rho}}\left(\frac{\psi}{\rho}\right)^{\frac{\rho}{\rho+2}}. \end{aligned} \tag{2.50}$$

That is

$$y(h_0) = (\rho+2)\left(\frac{\xi}{2}\right)^{\frac{2}{2+\rho}}\left(\frac{\psi}{\rho}\right)^{\frac{\rho}{\rho+2}}. \tag{2.51}$$

Therefore it holds for $2 < \nu < 3$

$$\begin{aligned} &\sup_{a\in\mathbb{R}_+}\left|D_a^{\nu-2}\varphi(a)\right| \\ &\leq \nu\left(\left(1+2^{1-\nu}\right)\Gamma(\nu-1)\|\varphi\|_{\infty,\mathbb{R}_+}\right)^{\frac{2}{\nu}}\left(\frac{6}{\nu(\nu-1)(\nu-2)}\sup_{a\in\mathbb{R}_+}\|D_a^\nu\varphi\|_{\infty,\mathbb{R}_+}\right)^{\frac{\nu-2}{\nu}} \qquad (2.52) \\ &= \left(\frac{6}{(\nu-1)(\nu-2)}\right)^{\frac{\nu-2}{\nu}}\left(\nu\left(1+2^{1-\nu}\right)\Gamma(\nu-1)\right)^{\frac{2}{\nu}}\|\varphi\|_{\infty,\mathbb{R}_+}^{\frac{2}{\nu}}\left(\sup_{a\in\mathbb{R}_+}\|D_a^\nu\varphi\|_{\infty,\mathbb{R}_+}\right)^{\frac{\nu-2}{\nu}}. \end{aligned}$$

The theorem is proved. □

Next we give right Riemann-Liouville fractional Landau type inequalities.

Convention 2.4 *Define* $D_{b-}^{\nu}\varphi(x) := 0$*, for any* $x > b$*, for any* $\nu > 0$, $\nu \notin \mathbb{N}$.

We present

Theorem 2.5 *Here* $1 < \nu < 2$ *and* $\varphi \in C_B(\mathbb{R}_-)$ *and* $D_{b-}^{\nu}\varphi \in C_B((-\infty, b])$, $\forall\, b \in \mathbb{R}_-$*, where* C_B *means continuous and bounded functions. We assume that* $\sup_{b\in\mathbb{R}_-}\left\|D_{b-}^{\nu}\varphi\right\|_{\infty,\mathbb{R}_-} < +\infty$*. Then*

$$\sup_{b\in\mathbb{R}_-}\left|D_{b-}^{\nu-1}\varphi(b)\right| \leq \left(\frac{\Gamma(\nu+1)}{(\nu-1)^{\nu-1}}\right)^{\frac{1}{\nu}}\|\varphi\|_{\infty,\mathbb{R}_-}^{\frac{1}{\nu}}\left(\sup_{b\in\mathbb{R}_-}\left\|D_{b-}^{\nu}\varphi\right\|_{\infty,\mathbb{R}_-}\right)^{\frac{\nu-1}{\nu}}, \tag{2.53}$$

a sharp inequality with a precisely determined constant. That is $\sup_{b\in\mathbb{R}_-}\left|D_{b-}^{\nu-1}\varphi(b)\right| < +\infty$.

Proof. For any $a \in (-\infty, b)$ we get (by Theorem 1.3)

$$\varphi(a) = \frac{D_{b-}^{\nu-1}\varphi(b)}{\Gamma(\nu)}(b-a)^{\nu-1} + \frac{D_{b-}^{\nu-2}\varphi(b)}{\Gamma(\nu-1)}(b-a)^{\nu-2} + \frac{1}{\Gamma(\nu)}\int_a^b (t-a)^{\nu-1}D_{b-}^{\nu}\varphi(t)\,dt. \tag{2.54}$$

Notice that ($x \in (-\infty, b]$)

$$D_{b-}^{\nu-2}\varphi(x) = D_{b-}^{-(2-\nu)}\varphi(x) = \frac{1}{\Gamma(2-\nu)}\int_x^b (t-x)^{1-\nu}\varphi(t)\,\mathrm{d}t,$$

and

$$\begin{aligned}\left|D_{b-}^{\nu-2}\varphi(x)\right| &\leq \frac{1}{\Gamma(2-\nu)}\int_x^b (t-x)^{1-\nu}|\varphi(t)|\,\mathrm{d}t \\ &\leq \frac{\|\varphi\|_{\infty,(-\infty,b]}}{\Gamma(2-\nu)}\int_x^b (t-x)^{1-\nu}\,\mathrm{d}t = \frac{\|\varphi\|_{\infty,(-\infty,b]}}{\Gamma(3-\nu)}(b-x)^{2-\nu}. \end{aligned} \quad (2.55)$$

That is

$$\left|D_{b-}^{\nu-2}\varphi(x)\right| \leq \frac{\|\varphi\|_{\infty,(-\infty,b]}}{\Gamma(3-\nu)}(b-x)^{2-\nu}, \quad \forall x \in (-\infty, b]. \quad (2.56)$$

In particular we have $D_{b-}^{\nu-2}\varphi(b) = 0$.

Hence by (2.54) we get

$$\varphi(a) = \frac{D_{b-}^{\nu-1}\varphi(b)}{\Gamma(\nu)}(b-a)^{\nu-1} + \frac{1}{\Gamma(\nu)}\int_a^b (t-a)^{\nu-1}D_{b-}^{\nu}\varphi(t)\,\mathrm{d}t, \quad \forall a \in (-\infty, b). \quad (2.57)$$

So that

$$\frac{D_{b-}^{\nu-1}\varphi(b)}{\Gamma(\nu)}(b-a)^{\nu-1} = \varphi(a) - \frac{1}{\Gamma(\nu)}\int_a^b (t-a)^{\nu-1}D_{b-}^{\nu}\varphi(t)\,\mathrm{d}t, \quad \forall a \in (-\infty, b). \quad (2.58)$$

Hence (set $h := b - a > 0$)

$$\begin{aligned}&\left|D_{b-}^{\nu-1}\varphi(b)\right| h^{\nu-1} = \left|\Gamma(\nu)\varphi(a) - \int_a^b (t-a)^{\nu-1}D_{b-}^{\nu}\varphi(t)\,\mathrm{d}t\right| \\ &\leq \Gamma(\nu)|\varphi(a)| + \int_a^b (t-a)^{\nu-1}\left|D_{b-}^{\nu}\varphi(t)\right|\mathrm{d}t \\ &\leq \Gamma(\nu)\|\varphi\|_{\infty,\mathbb{R}_-} + \left\|D_{b-}^{\nu}\varphi\right\|_{\infty,(-\infty,b]}\int_a^b (t-a)^{\nu-1}\,\mathrm{d}t \\ &= \Gamma(\nu)\|\varphi\|_{\infty,\mathbb{R}_-} + \left\|D_{b-}^{\nu}\varphi\right\|_{\infty,(-\infty,b]}\frac{(b-a)^{\nu}}{\nu} = \Gamma(\nu)\|\varphi\|_{\infty,\mathbb{R}_-} + \left\|D_{b-}^{\nu}\varphi\right\|_{\infty,(-\infty,b]}\frac{h^{\nu}}{\nu}. \end{aligned} \quad (2.59)$$

That is

$$\begin{aligned}\left|D_{b-}^{\nu-1}\varphi(b)\right| h^{\nu-1} &\leq \Gamma(\nu)\|\varphi\|_{\infty,\mathbb{R}_-} + \left\|D_{b-}^{\nu}\varphi\right\|_{\infty,(-\infty,b]}\frac{h^{\nu}}{\nu} \\ &\leq \Gamma(\nu)\|\varphi\|_{\infty,\mathbb{R}_-} + \frac{h^{\nu}}{\nu}\sup_{b\in\mathbb{R}_-}\left\|D_{b-}^{\nu}\varphi\right\|_{\infty,(-\infty,b]}. \end{aligned} \quad (2.60)$$

Furthermore we see that

$$\left|D_{b-}^{\nu-1}\varphi(b)\right| \leq \frac{\Gamma(\nu)\|\varphi\|_{\infty,\mathbb{R}_-}}{h^{\nu-1}} + \frac{h}{\nu}\sup_{b\in\mathbb{R}_-}\left\|D_{b-}^{\nu}\varphi\right\|_{\infty,(-\infty,b]}, \quad \forall h > 0 \text{ and } \forall b \in \mathbb{R}_-. \quad (2.61)$$

Consequently it holds

$$\sup_{b\in\mathbb{R}_-}\left|D_{b-}^{\nu-1}\varphi(b)\right| \leq \frac{\Gamma(\nu)\|\varphi\|_{\infty,\mathbb{R}_-}}{h^{\nu-1}} + \frac{h}{\nu}\sup_{b\in\mathbb{R}_-}\left\|D_{b-}^{\nu}\varphi\right\|_{\infty,\mathbb{R}_-}, \quad \forall h>0. \tag{2.62}$$

Call

$$\mu := \Gamma(\nu)\|\varphi\|_{\infty,\mathbb{R}_-}, \tag{2.63}$$

$$\theta := \frac{1}{\nu}\sup_{b\in\mathbb{R}_-}\left\|D_{b-}^{\nu}\varphi\right\|_{\infty,\mathbb{R}_-}, \tag{2.64}$$

both are greater than zero.

Here $1<\nu<2$ and $0<\nu-1<1$. Set also $\rho:=\nu-1$, *i.e.* $0<\rho<1$.

We consider the function

$$y(h) := \mu h^{-\rho} + \theta h, \ \ \forall\, h>0. \tag{2.65}$$

As in the proof of Theorem 2.2, y has a global minimum over $\mathbb{R}_+$ which is

$$y(h_0) = \frac{\rho+1}{\rho^{\frac{\rho}{\rho+1}}}\theta^{\frac{\rho}{\rho+1}}\mu^{\frac{1}{\rho+1}}, \tag{2.66}$$

where the critical number

$$h_0 = \left(\frac{\rho\mu}{\theta}\right)^{\frac{1}{\rho+1}} > 0.$$

Consequently we derive that

$$\sup_{b\in\mathbb{R}_-}\left|D_{b-}^{\nu-1}\varphi(b)\right| \leq \left(\frac{\nu}{(\nu-1)^{\frac{\nu-1}{\nu}}}\right)\left(\frac{1}{\nu}\sup_{b\in\mathbb{R}_-}\left\|D_{b-}^{\nu}\varphi\right\|_{\infty,\mathbb{R}_-}\right)^{\frac{\nu-1}{\nu}}\left(\Gamma(\nu)\|\varphi\|_{\infty,\mathbb{R}_-}\right)^{\frac{1}{\nu}}$$
$$= \left(\frac{\Gamma(\nu+1)}{(\nu-1)^{\nu-1}}\right)^{\frac{1}{\nu}}\|\varphi\|_{\infty,\mathbb{R}_-}^{\frac{1}{\nu}}\left(\sup_{b\in\mathbb{R}_-}\left\|D_{b-}^{\nu}\varphi\right\|_{\infty,\mathbb{R}_-}\right)^{\frac{\nu-1}{\nu}}, \text{for } 1<\nu<2. \tag{2.67}$$

The theorem is proved. □

We continue with

Theorem 2.6 *Here* $2<\nu<3$ *and* $\varphi\in C_B(\mathbb{R}_-)$ *and* $D_{b-}^{\nu}\varphi\in C_B((-\infty,b])$, $\forall\, b\in\mathbb{R}_-$. *We assume that* $\sup_{b\in\mathbb{R}_-}\left\|D_{b-}^{\nu}\varphi\right\|_{\infty,\mathbb{R}_-}<+\infty$. *Then*

i)

$$\sup_{b\in\mathbb{R}_-}\left|D_{b-}^{\nu-1}\varphi(b)\right|$$
$$\leq \left(\frac{5}{\nu-1}\right)^{\frac{\nu-1}{\nu}}\left(\nu\left(1+2^{2-\nu}\right)\Gamma(\nu)\right)^{\frac{1}{\nu}}\|\varphi\|_{\infty,\mathbb{R}_-}^{\frac{1}{\nu}}\left(\sup_{b\in\mathbb{R}_-}\left\|D_{b-}^{\nu}\varphi\right\|_{\infty,\mathbb{R}_-}\right)^{\frac{\nu-1}{\nu}}, \tag{2.68}$$

ii)

$$\sup_{b\in\mathbb{R}_-}\left|D_{b-}^{\nu-2}\varphi(b)\right| \tag{2.69}$$
$$\leq\left(\frac{6}{(\nu-1)(\nu-2)}\right)^{\frac{\nu-2}{\nu}}\left(\nu\left(1+2^{1-\nu}\right)\Gamma(\nu-1)\right)^{\frac{2}{\nu}}\|\varphi\|_{\infty,\mathbb{R}_-}^{\frac{2}{\nu}}\left(\sup_{b\in\mathbb{R}_-}\left\|D_{b-}^{\nu}\varphi\right\|_{\infty,\mathbb{R}_-}\right)^{\frac{\nu-2}{\nu}}.$$

The above inequalities are nearly sharp with precisely determined constants.

That is $\sup_{b\in\mathbb{R}_-}\left|D_{b-}^{\nu-1}\varphi(b)\right|,\ \sup_{b\in\mathbb{R}_-}\left|D_{b-}^{\nu-2}\varphi(b)\right|<+\infty.$

Proof. For any $a\in(-\infty,b)$ we get (by Theorem 1.3)

$$\varphi(a)=\frac{D_{b-}^{\nu-1}\varphi(b)}{\Gamma(\nu)}(b-a)^{\nu-1}+\frac{D_{b-}^{\nu-2}\varphi(b)}{\Gamma(\nu-1)}(b-a)^{\nu-2}$$
$$+\frac{D_{b-}^{\nu-3}\varphi(b)}{\Gamma(\nu-2)}(b-a)^{\nu-3}+\frac{1}{\Gamma(\nu)}\int_a^b(t-a)^{\nu-1}D_{b-}^{\nu}\varphi(t)\,\mathrm{d}t. \tag{2.70}$$

Notice that ($x\in(-\infty,b]$)

$$D_{b-}^{\nu-3}\varphi(x)=D_{b-}^{-(3-\nu)}\varphi(x)=\frac{1}{\Gamma(3-\nu)}\int_x^b(t-x)^{2-\nu}\varphi(t)\,\mathrm{d}t,$$

and

$$\left|D_{b-}^{\nu-3}\varphi(x)\right|\leq\frac{\|\varphi\|_{\infty,(-\infty,b]}}{\Gamma(4-\nu)}(b-x)^{3-\nu},\quad\forall x\in(-\infty,b]. \tag{2.71}$$

That is $D_{b-}^{\nu-3}\varphi(b)=0$.

Hence by (2.70), for all $a\in(-\infty,b)$, we get

$$\frac{D_{b-}^{\nu-1}\varphi(b)}{\Gamma(\nu)}(b-a)^{\nu-1}+\frac{D_{b-}^{\nu-2}\varphi(b)}{\Gamma(\nu-1)}(b-a)^{\nu-2}=\varphi(a)-\frac{1}{\Gamma(\nu)}\int_a^b(t-a)^{\nu-1}D_{b-}^{\nu}\varphi(t)\,\mathrm{d}t, \tag{2.72}$$

Let $h>0:b-a=h$, *i.e.* $b=a+h$, then

$$\frac{D_{b-}^{\nu-1}\varphi(b)}{\Gamma(\nu)}h^{\nu-1}+\frac{D_{b-}^{\nu-2}\varphi(b)}{\Gamma(\nu-1)}h^{\nu-2}=\varphi(a)-\frac{1}{\Gamma(\nu)}\int_a^{a+h}(t-a)^{\nu-1}D_{b-}^{\nu}\varphi(t)\,\mathrm{d}t=:\overline{A}. \tag{2.73}$$

For $b-a=2h$, *i.e.* $b=a+2h$, we get

$$\frac{D_{b-}^{\nu-1}\varphi(b)}{\Gamma(\nu)}2^{\nu-1}h^{\nu-1}+\frac{D_{b-}^{\nu-2}\varphi(b)}{\Gamma(\nu-1)}2^{\nu-2}h^{\nu-2}$$
$$=\varphi(a)-\frac{1}{\Gamma(\nu)}\int_a^{a+2h}(t-a)^{\nu-1}D_{b-}^{\nu}\varphi(t)\,\mathrm{d}t=:\overline{B}. \tag{2.74}$$

We solve the system of two equations (2.73), (2.74) for two unknowns $D_{b-}^{\nu-1}\varphi(b)$ and $D_{b-}^{\nu-2}\varphi(b)$. As in the proof of Theorem 2.3, we find

$$D_{b-}^{\nu-1}\varphi(b)=\frac{\Gamma(\nu)\left(\overline{B}-2^{\nu-2}\overline{A}\right)}{2^{\nu-2}h^{\nu-1}}, \tag{2.75}$$

and

$$D_{b-}^{\nu-2}\varphi(b)=\frac{\Gamma(\nu-1)\left(2^{\nu-1}\overline{A}-\overline{B}\right)}{2^{\nu-2}h^{\nu-2}}. \tag{2.76}$$

We notice that

$$\left|\overline{A}\right|\leq\|\varphi\|_{\infty,\mathbb{R}_-}+\frac{h^{\nu}}{\Gamma(\nu+1)}\sup_{b\in\mathbb{R}_-}\left\|D_{b-}^{\nu}\varphi\right\|_{\infty,\mathbb{R}_-}, \tag{2.77}$$

and

$$\left|\overline{B}\right|\leq\|\varphi\|_{\infty,\mathbb{R}_-}+\frac{2^{\nu}h^{\nu}}{\Gamma(\nu+1)}\sup_{b\in\mathbb{R}_-}\left\|D_{b-}^{\nu}\varphi\right\|_{\infty,\mathbb{R}_-}. \tag{2.78}$$

As in the proof of Theorem 2.3, we obtain

$$\sup_{b\in\mathbb{R}_-}\left|D_{b-}^{\nu-1}\varphi(b)\right|\leq\frac{\left(1+2^{2-\nu}\right)\Gamma(\nu)\|\varphi\|_{\infty,\mathbb{R}_-}}{h^{\nu-1}}+\frac{5h}{\nu}\sup_{b\in\mathbb{R}_-}\left\|D_{b-}^{\nu}\varphi\right\|_{\infty,\mathbb{R}_-}, \tag{2.79}$$

and for all $h>0,\ 2<\nu<3$

$$\sup_{b\in\mathbb{R}_-}\left|D_{b-}^{\nu-2}\varphi(b)\right|\leq\frac{2\left(1+2^{1-\nu}\right)\Gamma(\nu-1)\|\varphi\|_{\infty,\mathbb{R}_-}}{h^{\nu-2}}+\frac{6h^2}{\nu(\nu-1)}\sup_{b\in\mathbb{R}_-}\left\|D_{b-}^{\nu}\varphi\right\|_{\infty,\mathbb{R}_-}. \tag{2.80}$$

As in the proof of Theorem 2.3, we find similarly optimal upper bounds in (2.79) and (2.80).

We derive

$$\begin{aligned}&\sup_{b\in\mathbb{R}_-}\left|D_{b-}^{\nu-1}\varphi(b)\right|\\&\leq\frac{\nu}{(\nu-1)^{\frac{\nu-1}{\nu}}}\left(\frac{5}{\nu}\sup_{b\in\mathbb{R}_-}\left\|D_{b-}^{\nu}\varphi\right\|_{\infty,\mathbb{R}_-}\right)^{\frac{\nu-1}{\nu}}\left(\left(1+2^{2-\nu}\right)\Gamma(\nu)\|\varphi\|_{\infty,\mathbb{R}_-}\right)^{\frac{1}{\nu}}\\&=\left(\frac{5}{\nu-1}\right)^{\frac{\nu-1}{\nu}}\left(\nu\left(1+2^{2-\nu}\right)\Gamma(\nu)\right)^{\frac{1}{\nu}}\|\varphi\|_{\infty,\mathbb{R}_-}^{\frac{1}{\nu}}\left(\sup_{b\in\mathbb{R}_-}\left\|D_{b-}^{\nu}\varphi\right\|_{\infty,\mathbb{R}_-}\right)^{\frac{\nu-1}{\nu}},\end{aligned} \tag{2.81}$$

and for $2<\nu<3$

$$\begin{aligned}&\sup_{b\in\mathbb{R}_-}\left|D_{b-}^{\nu-2}\varphi(b)\right|\\&\leq\nu\left(\left(1+2^{1-\nu}\right)\Gamma(\nu-1)\|\varphi\|_{\infty,\mathbb{R}_-}\right)^{\frac{2}{\nu}}\left(\frac{6}{\nu(\nu-1)(\nu-2)}\sup_{b\in\mathbb{R}_-}\left\|D_{b-}^{\nu}\varphi\right\|_{\infty,\mathbb{R}_-}\right)^{\frac{\nu-2}{\nu}}\\&=\left(\frac{6}{(\nu-1)(\nu-2)}\right)^{\frac{\nu-2}{\nu}}\left(\nu\left(1+2^{1-\nu}\right)\Gamma(\nu-1)\right)^{\frac{2}{\nu}}\|\varphi\|_{\infty,\mathbb{R}_-}^{\frac{2}{\nu}}\left(\sup_{b\in\mathbb{R}_-}\left\|D_{b-}^{\nu}\varphi\right\|_{\infty,\mathbb{R}_-}\right)^{\frac{\nu-2}{\nu}}.\end{aligned} \tag{2.82}$$

The theorem is proved. □

An application of Theorem 2.2 follows ($\nu=1.5$).

Corollary 2.7 *Let $\varphi \in C_B(\mathbb{R}_+)$ such that $D_a^{1.5}\varphi \in C_B([a,+\infty))$, $\forall\, a \in \mathbb{R}_+$. Assume that $\sup_{a\in\mathbb{R}_+} \left\| D_a^{1.5}\varphi \right\|_{\infty,\mathbb{R}_+} < +\infty$. Then*

$$\sup_{a\in\mathbb{R}_+} \left| D_a^{0.5}\varphi(a) \right| \leq 1.52323665 \left\|\varphi\right\|_{\infty,\mathbb{R}_+}^{0.666666667} \left(\sup_{a\in\mathbb{R}_+} \left\| D_a^{1.5}\varphi \right\|_{\infty,\mathbb{R}_+} \right)^{0.333333333}, \tag{2.83}$$

a sharp inequality. That is $\sup_{a\in\mathbb{R}_+} \left| D_a^{0.5}\varphi(a) \right| < +\infty$.

An application of Theorem 2.5 follows ($\nu = 1.5$).

Corollary 2.8 *Let $\varphi \in C_B(\mathbb{R}_-)$ such that $D_{b-}^{1.5}\varphi \in C_B((-\infty,b])$, $\forall\, b \in \mathbb{R}_-$. Assume that $\sup_{b\in\mathbb{R}_-} \left\| D_{b-}^{1.5}\varphi \right\|_{\infty,\mathbb{R}_-} < +\infty$. Then*

$$\sup_{b\in\mathbb{R}_-} \left| D_{b-}^{0.5}\varphi(b) \right| \leq 1.52323665 \left\|\varphi\right\|_{\infty,\mathbb{R}_-}^{0.666666667} \left(\sup_{b\in\mathbb{R}_-} \left\| D_{b-}^{1.5}\varphi \right\|_{\infty,\mathbb{R}_-} \right)^{0.333333333}, \tag{2.84}$$

a sharp inequality. That is $\sup_{b\in\mathbb{R}_-} \left| D_{b-}^{0.5}\varphi(b) \right| < +\infty$.

3 Appendix

We give the following simplified fractional Taylor formulae.

Corollary 3.1 *Let $\nu > 0$, $\nu \notin \mathbb{N}$ and $m = \lceil \nu \rceil$; φ, $D_a^{\nu}\varphi \in C([a,b])$. Then*

$$\varphi(b) = \sum_{j=1}^{m-1} \frac{D_a^{\nu-j}\varphi(a)}{\Gamma(\nu-j+1)} (b-a)^{\nu-j} + \frac{1}{\Gamma(\nu)} \int_a^b (b-t)^{\nu-1} D_a^{\nu}\varphi(t)\, dt. \tag{3.1}$$

Proof. It is $m > \nu$, hence $m - \nu > 0$. We have that

$$D_a^{\nu-m}\varphi(x) = D_a^{-(m-\nu)}\varphi(x) = \frac{1}{\Gamma(m-\nu)} \int_a^x (x-t)^{m-\nu-1} \varphi(t)\, dt,$$

and

$$\begin{aligned} \left| D_a^{\nu-m}\varphi(x) \right| &\leq \frac{1}{\Gamma(m-\nu)} \int_a^x (x-t)^{m-\nu-1} \left|\varphi(t)\right| dt \\ &\leq \frac{\left\|\varphi\right\|_{\infty,[a,b]}}{\Gamma(m-\nu)} \int_a^x (x-t)^{m-\nu-1}\, dt = \frac{(x-a)^{m-\nu} \left\|\varphi\right\|_{\infty,[a,b]}}{(m-\nu)\Gamma(m-\nu)} \\ &= \frac{\left\|\varphi\right\|_{\infty,[a,b]}}{\Gamma(m-\nu+1)} (x-a)^{m-\nu}. \end{aligned} \tag{3.2}$$

That is

$$\left| D_a^{\nu-m}\varphi(x) \right| \leq \frac{\left\|\varphi\right\|_{\infty,[a,b]}}{\Gamma(m-\nu+1)} (x-a)^{m-\nu}, \quad \forall x \in [a,b]. \tag{3.3}$$

Hence $D_{a}^{\nu-m}\varphi(a)=0$.

Use also Theorem 1.2. □

Corollary 3.2 *Let $\nu>0$, $\nu\notin\mathbb{N}$ and $m=\lceil\nu\rceil$; φ, $D_{b-}^{\nu}\varphi\in C([a,b])$. Then*

$$\varphi(a)=\sum_{j=1}^{m-1}\frac{D_{b-}^{\nu-j}\varphi(b)}{\Gamma(\nu-j+1)}(b-a)^{\nu-j}+\frac{1}{\Gamma(\nu)}\int_{a}^{b}(t-a)^{\nu-1}D_{b-}^{\nu}\varphi(t)\,\mathrm{d}t. \tag{3.4}$$

Proof. We have that

$$D_{b-}^{\nu-m}\varphi(x)=D_{b-}^{-(m-\nu)}\varphi(x)=\frac{1}{\Gamma(m-\nu)}\int_{x}^{b}(t-x)^{m-\nu-1}\varphi(t)\,\mathrm{d}t,$$

and

$$\left|D_{b-}^{\nu-m}\varphi(x)\right|\leq\frac{\|\varphi\|_{\infty,[a,b]}}{\Gamma(m-\nu+1)}(b-x)^{m-\nu},\quad\forall x\in[a,b]. \tag{3.5}$$

Hence $D_{b-}^{\nu-m}\varphi(b)=0$. Use also Theorem 1.3. □

We give also an interesting new fractional inequality of Ostrowski like.

Theorem 3.3 *Let $\nu>0$, $\nu\notin\mathbb{N}$ and $m=\lceil\nu\rceil$, with $\varphi\in C([a,b])$. Let also $x_0\in[a,b]$ be fixed. Assumed that $D_{x_0}^{\varphi}\in C([x_0,b])$ and $D_{x_0-}^{\nu}\varphi\in C([a,x_0])$. Then*

$$\begin{aligned}&\left|\int_{a}^{b}\varphi(x)\,\mathrm{d}x-\sum_{j=1}^{m-1}\frac{1}{\Gamma(\nu-j+2)}\left[D_{x_0}^{\nu-j}\varphi(x_0)(b-x_0)^{\nu-j+1}+D_{x_0-}^{\nu-j}\varphi(x_0)(x_0-a)^{\nu-j+1}\right]\right|\\&\leq\frac{1}{\Gamma(\nu+2)}\left[\left\|D_{x_0}^{\nu}\varphi\right\|_{\infty,[x_0,b]}(b-x_0)^{\nu+1}+\left\|D_{x_0-}^{\nu}\varphi\right\|_{\infty,[a,x_0]}(x_0-a)^{\nu+1}\right]\\&\leq\frac{\max\left\{\left\|D_{x_0-}^{\nu}\varphi\right\|_{\infty,[a,x_0]},\left\|D_{x_0}^{\nu}\varphi\right\|_{\infty,[x_0,b]}\right\}}{\Gamma(\nu+2)}\left[(b-x_0)^{\nu+1}+(x_0-a)^{\nu+1}\right].\end{aligned} \tag{3.6}$$

Proof. We have that

$$\varphi(x)=\sum_{j=1}^{m-1}\frac{D_{x_0}^{\nu-j}\varphi(x_0)}{\Gamma(\nu-j+1)}(x-x_0)^{\nu-j}+\frac{1}{\Gamma(\nu)}\int_{x_0}^{x}(x-t)^{\nu-1}D_{x_0}^{\nu}\varphi(t)\,\mathrm{d}t,\ \forall x\in[x_0,b]. \tag{3.7}$$

Also we have

$$\varphi(x)=\sum_{j=1}^{m-1}\frac{D_{x_0-}^{\nu-j}\varphi(x_0)}{\Gamma(\nu-j+1)}(x_0-x)^{\nu-j}+\frac{1}{\Gamma(\nu)}\int_{x}^{x_0}(t-x)^{\nu-1}D_{x_0-}^{\nu}\varphi(t)\,\mathrm{d}t,\ \forall x\in[a,x_0]. \tag{3.8}$$

Thus

$$\varphi(x)-\sum_{j=1}^{m-1}\frac{D_{x_0}^{\nu-j}\varphi(x_0)}{\Gamma(\nu-j+1)}(x-x_0)^{\nu-j}=\frac{1}{\Gamma(\nu)}\int_{x_0}^{x}(x-t)^{\nu-1}D_{x_0}^{\nu}\varphi(t)\,\mathrm{d}t,\ \forall x\in[x_0,b], \tag{3.9}$$

and

$$\varphi(x)-\sum_{j=1}^{m-1}\frac{D_{x_0-}^{\nu-j}\varphi(x_0)}{\Gamma(\nu-j+1)}(x_0-x)^{\nu-j}=\frac{1}{\Gamma(\nu)}\int_x^{x_0}(t-x)^{\nu-1}D_{x_0-}^{\nu}\varphi(t)\,\mathrm{d}t,\ \forall x\in[a,x_0]. \quad (3.10)$$

Therefore we get

$$\int_{x_0}^{b}\varphi(x)\,\mathrm{d}x-\sum_{j=1}^{m-1}\frac{D_{x_0}^{\nu-j}\varphi(x_0)}{\Gamma(\nu-j+2)}(b-x_0)^{\nu-j+1}=\frac{1}{\Gamma(\nu)}\int_{x_0}^{b}\int_{x_0}^{x}(x-t)^{\nu-1}D_{x_0}^{\nu}\varphi(t)\,\mathrm{d}t\,\mathrm{d}x \quad (3.11)$$

and

$$\int_{a}^{x_0}\varphi(x)\,\mathrm{d}x-\sum_{j=1}^{m-1}\frac{D_{x_0-}^{\nu-j}\varphi(x_0)}{\Gamma(\nu-j+2)}(x_0-a)^{\nu-j+1}=\frac{1}{\Gamma(\nu)}\int_a^{x_0}\int_x^{x_0}(t-x)^{\nu-1}D_{x_0-}^{\nu}\varphi(t)\,\mathrm{d}t\,\mathrm{d}x. \quad (3.12)$$

Adding (3.11) and (3.12) we derive

$$E:=\int_a^b\varphi(x)\,\mathrm{d}x-\sum_{j=1}^{m-1}\frac{1}{\Gamma(\nu-j+2)}\left[D_{x_0}^{\nu-j}\varphi(x_0)(b-x_0)^{\nu-j+1}+D_{x_0-}^{\nu-j}\varphi(x_0)(x_0-a)^{\nu-j+1}\right]$$
$$=\frac{1}{\Gamma(\nu)}\left[\int_{x_0}^{b}\int_{x_0}^{x}(x-t)^{\nu-1}D_{x_0}^{\nu}\varphi(t)\,\mathrm{d}t\,\mathrm{d}x+\int_a^{x_0}\int_x^{x_0}(t-x)^{\nu-1}D_{x_0-}^{\nu}\varphi(t)\,\mathrm{d}t\,\mathrm{d}x\right]. \quad (3.13)$$

Therefore we obtain

$$|E|\le\frac{1}{\Gamma(\nu)}\left[\int_{x_0}^{b}\int_{x_0}^{x}(x-t)^{\nu-1}\left|D_{x_0}^{\nu}\varphi(t)\right|\,\mathrm{d}t\,\mathrm{d}x+\int_a^{x_0}\int_x^{x_0}(t-x)^{\nu-1}\left|D_{x_0-}^{\nu}\varphi(t)\right|\,\mathrm{d}t\,\mathrm{d}x\right]$$
$$\le\frac{1}{\Gamma(\nu)}\left[\left\|D_{x_0}^{\nu}\varphi\right\|_{\infty,[x_0,b]}\int_{x_0}^{b}\int_{x_0}^{x}(x-t)^{\nu-1}\,\mathrm{d}t\,\mathrm{d}x+\left\|D_{x_0-}^{\nu}\varphi\right\|_{\infty,[a,x_0]}\int_a^{x_0}\int_x^{x_0}(t-x)^{\nu-1}\,\mathrm{d}t\,\mathrm{d}x\right]$$
$$=\frac{1}{\Gamma(\nu+2)}\left[\left\|D_{x_0}^{\nu}\varphi\right\|_{\infty,[x_0,b]}(b-x_0)^{\nu+1}+\left\|D_{x_0-}^{\nu}\varphi\right\|_{\infty,[a,x_0]}(x_0-a)^{\nu+1}\right]$$
$$\le\frac{\max\left\{\left\|D_{x_0-}^{\nu}\varphi\right\|_{\infty,[a,x_0]},\left\|D_{x_0}^{\nu}\varphi\right\|_{\infty,[x_0,b]}\right\}}{\Gamma(\nu+2)}\left[(b-x_0)^{\nu+1}+(x_0-a)^{\nu+1}\right]. \quad (3.14)$$

□

We finish with

Corollary 3.4 *Assume all as in Theorem 3.3 with* $1<\nu<2$. *Then*

$$\left|\int_a^b\varphi(x)\,\mathrm{d}x-\frac{1}{\Gamma(\nu+1)}\left[D_{x_0}^{\nu-1}\varphi(x_0)(b-x_0)^{\nu}+D_{x_0-}^{\nu-1}\varphi(x_0)(x_0-a)^{\nu}\right]\right|$$
$$\le\frac{1}{\Gamma(\nu+2)}\left[\left\|D_{x_0}^{\nu}\varphi\right\|_{\infty,[x_0,b]}(b-x_0)^{\nu+1}+\left\|D_{x_0-}^{\nu}\varphi\right\|_{\infty,[a,x_0]}(x_0-a)^{\nu+1}\right]$$
$$\le\max\left\{\left\|D_{x_0-}^{\nu}\varphi\right\|_{\infty,[a,x_0]},\left\|D_{x_0}^{\nu}\varphi\right\|_{\infty,[x_0,b]}\right\}\frac{(b-x_0)^{\nu+1}+(x_0-a)^{\nu+1}}{\Gamma(\nu+2)}. \quad (3.15)$$

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