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BRANCHED HAMILTONIANS FOR QUADRATIC TYPE LIÉNARD OSCILLATOR

Bijan Bagchi*
Department of Physics, Shiv Nadar University, Dadri, UP 201314, India

Dibyendu Ghosh†
Department of Mathematics, Acharya Prafulla Chandra College,
New Barrackpur, West Bengal, 24 Parganas 700131, India

Tarun R. Tummuru‡
Department of Physics, Shiv Nadar University, Dadri, UP 201314, India

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Abstract. We point out that when a quadratic type Liénard equation is suitably interpreted shows branching due to the double valuedness of the governing Hamiltonian. Under certain approximation of the guiding coupling constant we derive its quantum counterpart that is guided by a momentum-dependent mass function.

Keywords: Branched Hamiltonians, nonstandard Lagrangian, Liénard oscillator, momentum dependent mass.

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1 Introduction

Branched dynamical systems have been a focus of attention of late [1, 2, 3, 4, 5, 6, 7, 8, 9] after an inquiry carried out by Shapere and Wilczek [1] in this direction. A couple of notable papers following this work include the one of Curtright and Zachos [2] who looked at a classical system as

*e-mail address: bbagchi123@gmail.com
†e-mail address: dghoshbrsc@gmail.com
‡e-mail address: tt295@snu.edu.in

given by a pair of convex, smoothly tied functions of the velocity variable v possessing a quantum counterpart that exhibits a double-valued Hamiltonian resembling a supersymmetric system in the momentum space, and that of Bagchi *et al.* [3, 4] who explored a couple of nonlinear models governed by a set of Hamiltonians of Liénard type [10] in which the velocity variable appears linearly that yield to branching.

In this communication we aim to carry out our investigation further for the Mathews and Lakshmanan's model of a nonlinear oscillator describing quasi-harmonic oscillations which admits of a Lagrangian interpretation [11]. Note that the ensuing differential equation yields to simple harmonic bound state solutions.

Our plan is as follows:

In the following section we give a brief review of branched Hamiltonians that arise in the context of the Liénard class of linear differential equation having a Lagrangian interpretation; subsequently we propose a new branching model for a modified Liénard equation consisting of a quadratic velocity term and which too admits of a Lagrangian. Identifying a small coupling parameter and expanding the Hamiltonian in terms of it shows that a momentum-dependent effective mass quantum system can be realized. Finally a summary of our scheme is given.

2 Branched Hamiltonians for the linear Liénard equation: A brief review

An attempt to track down branched Hamiltonians in the context of nonlinear differential equations was first done for the linear class of Liénard type of models [3, 4]. To review briefly the approach let us note that Liénard equation when the velocity variable appears linearly is typically given by the form

$$\ddot{x} + g(x)\dot{x} + h(x) = 0 \tag{2.1}$$

where $g(x)$ and $h(x)$ are arbitrary continuously differentiable functions depending on the spatial coordinate x and an overdot stands for a derivative with respect to the time variable.

We concentrate on the model of Mathews and Lakshmanan [10] which looks at the particular case wherein $g(x) = k\,x$ and $h(x) = \lambda\,x + \frac{k^2}{9}\,x^3$ (these are odd functions of x) and which yields the equation of motion

$$\ddot{x} + k\,x\,\dot{x} + \frac{k^2}{9}\,x^3 + \lambda\,x = 0\,, \quad k\,,\,\lambda > 0\,, \tag{2.2}$$

representing a cubic oscillator subject to a damped nonlinear force as indicated by the product term $k\,x\,\dot{x}$. This equation is appealing in that it can be seen to follow from a Lagrangian as provided by the precise form

$$L = \frac{27\,\lambda^3}{2\,k^2}(k\,\dot{x} + \frac{k^2\,x^2}{3} + 3\,\lambda)^{-1} + \frac{3\,\lambda\,\dot{x}}{2\,k} - \frac{9\,\lambda^2}{2\,k^2}\,. \tag{2.3}$$

After some straightforward manipulations the corresponding Hamiltonian H can also be written down

$$H_{(x\,,p)} = \frac{9\,\lambda^2}{2\,k^2}\left[2 - 2\left(1 - \frac{2\,k\,p}{3\,\lambda}\right)^{1/2} + \frac{k^2\,x^2}{9\,\lambda} - \frac{2\,k\,p}{3\,\lambda} - \frac{2\,k^3\,x^2\,p}{27\,\lambda^2}\right]. \tag{2.4}$$

One can see that H is of non-standard type: the coordinate and momentum are mixed so that the Hamiltonian cannot be written as the sum of individual kinetic and potential energy terms.

The Lagrangian (2.3), however, is not unique in producing the Hamiltonian $H_{(x\,,\,p)}$ as was pointed out in [3]. Towards this end Bagchi *et al.* [3] considered an entirely different form

$$L(x\,,\,v) = C(v + f(x))^{\frac{2\,m+1}{2\,m-1}} - \delta\,, \quad \text{where} \quad C = \frac{1-2\,m}{1+2\,m}\,\delta^{\frac{2}{1-2\,m}}\,, \tag{2.5}$$

where v is the velocity variable, $f(x)$ is an arbitrary smooth function, m is a non-negative integer and δ is some suitable constant quantity. The motivation for such a form came from the model of Curtright and Zachos [2] who were interested in seeking some concrete example of branched Hamiltonians. The present one of (2.5) differs from it in the presence of additional distinguishing features in the form of an inverse exponent in the first term and the inclusion of a general function term $f(x)$. Since L inevitably leads to multiple values for p, the resulting system possesses a branched structure

$$H_{\pm}(x\,,\,p) = (-p)f(x) - \frac{2\,\delta}{2\,m+1}(\pm\sqrt{-p})^{2\,m+1} + \delta\,. \tag{2.6}$$

Quite interestingly, for the specific case of $m = 0$, $H_{\pm}$ reduce to

$$H_{\pm} = (-p)f(x) \mp 2\,\delta\sqrt{-p} + \delta\,. \tag{2.7}$$

When one defines $f(x)$ and δ as

$$f(x) = \frac{\lambda}{2}\,x^2 + \frac{9\,\lambda^2}{2\,k^2}\,, \qquad \delta = \frac{9\,\lambda^2}{2\,k^2}\,, \tag{2.8}$$

then with the trivial shift $p \to \frac{2\,k}{3\,\lambda}p - 1$, we get

$$H_{\pm} = \frac{9\,\lambda^2}{2\,k^2}\left[2 \mp 2\left(1 - \frac{2\,k\,p}{3\,\lambda}\right)^{1/2} + \frac{k^2\,x^2}{9\,\lambda} - \frac{2\,k\,p}{3\,\lambda} - \frac{2\,k^3\,x^2\,p}{27\,\lambda^2}\right]. \tag{2.9}$$

Evidently, the positive branch corresponds to the form of $H_{(x\,,\,p)}$ in (2.3) while both $H_{\pm}$ reveal the presence of a linear harmonic potential in the limit $k \to 0$. The pair Hamiltonians above speak of branching in the momentum space as p deviates from the value $\frac{3\,\lambda}{2\,k}$.

3 A new branching model: Branched Hamiltonians for the quadratic Liénard equation

We now turn to the case of the quadratic Liénard equation. In this regard we consider the following Lagrangian

$$L = \frac{1}{2}\left(\frac{1}{1+\lambda\,x^2}\right)(\dot{x}^2 - \alpha\,x^2)\,, \quad \lambda\,,\,\alpha > 0\,. \tag{3.1}$$

Note that such a Lagrangian was also proposed by Mathews and Lakshmanan [11] as another viable example of a nonlinear oscillator. It is noteworthy that the above quadratic system is best with a number of interesting integrability properties. For a discussion of these we refer to [12].

Employing Lagrange's equation of motion we readily derive the the differential equation

$$(1+\lambda\, x^2)\ddot{x} - (\lambda\, x)\dot{x}^2 + \alpha\, x = 0\,, \quad \lambda > 0\,. \tag{3.2}$$

The presence of a quadratic velocity term is to be noted. Remarkably, the above equation admits of simple harmonic bound state solutions of the from

$$x = A\sin(\omega\, t + \phi) \tag{3.3}$$

with the frequency and amplitude restricted through

$$\omega^2 = \frac{\alpha}{1+\lambda\, A^2}\,. \tag{3.4}$$

In other words, the present scheme of the nonlinear oscillator with periodic solutions that qualify as having a simple harmonic form. The nonlinear equation is an interesting example of a system with so-called nonlinear quasi-harmonic oscillations. It has been proved recently [12] that this problem can be generalized to the two-dimensional case, and even to the n-dimensional case.

The above Lagrangian gives respectively for the momentum p and the accompanying Hamiltonian H the forms

$$p = \frac{\dot{x}}{(1+\lambda\, x^2)}\,, \tag{3.5}$$

$$H = \frac{p^2}{2}(1+\lambda\, x^2) + \frac{\alpha\, x^2}{2(1+\lambda\, x^2)}\,. \tag{3.6}$$

With this background the model we propose to study an extended scheme having the Lagrangian

$$L = \frac{\eta}{2(1+\lambda\, x^2)}\left[(v-\zeta)^r - \alpha\, x^2 + \zeta^2\right], \tag{3.7}$$

where the parameters $r\,,\ \eta,\ \lambda,\ \alpha$ and ζ are all real. The nonlinear equation of motion (3.2) is obtainable in the special case when $\eta = 1\,,\ r = 2$, and $\zeta = 0$.

On the other hand, if we want to make a connection with the Hamiltonian of Curtright and Zachos [2], the following choice of parameters $\eta = 6\left(\frac{1}{4}\right)^{\frac{2}{3}}\,,\ r = \frac{1}{3}$, and $\zeta = 1$ proves relevant. First of all, these give for the Lagrangian the form

$$L = \frac{3}{4^{\frac{2}{3}}(1+\lambda\, x^2)}\left[(v-1)^{\frac{1}{3}} - \alpha\, x^2 + 1\right], \tag{3.8}$$

which allows the momentum to be determined by the relation

$$p = \frac{\partial L}{\partial v} = \frac{(v-1)^{-\frac{2}{3}}}{4^{\frac{2}{3}}(1+\lambda\, x^2)}\,. \tag{3.9}$$

Secondly, we can solve for the velocity to get

$$v_{\pm}(p) = 1 \mp \frac{p^{-\frac{3}{2}}}{4(1+\lambda\, x^2)^{\frac{3}{2}}}\,, \tag{3.10}$$

which at once reveals a branching character. Observe that the velocity function depends on the spatial coordinate.

One thus arrives at a pair of branched Hamiltonian

$$H_{\pm} = p \pm \frac{p^{-\frac{1}{2}}}{2(1+\lambda\, x^2)^{\frac{3}{2}}} + \frac{3(\alpha\, x^2 - 1)}{4^{\frac{2}{3}}(1+\lambda\, x^2)}\ , \tag{3.11}$$

which coincide with the ones deduced in [2].

It is instructive to look at the behavior of the system for small values of the spatial coordinate x. A simple binomial expansion gives

$$H_{\pm} = p \pm \frac{1}{2}\, p^{-\frac{1}{2}}(1+\lambda\, x^2)^{-\frac{3}{2}} + \frac{3}{4^{\frac{2}{3}}}(\alpha\, x^2 - 1)(1+\lambda\, x^2)^{-1}\ , \tag{3.12}$$

which for small values of λ yields

$$H_{\pm} = -\left[\pm\frac{3}{4}\,\lambda\, p^{-\frac{1}{2}} - \frac{3}{4^{\frac{2}{3}}}(\alpha+\lambda)\right] x^2 + p \pm \frac{1}{2\sqrt{p}} - \frac{3}{4^{\frac{2}{3}}} + \mathcal{O}(x^4)\ . \tag{3.13}$$

Taking the Fourier transform to the momentum space and shifting the expression by the last constant term, we get a quantum form of the Hamiltonian

$$H_{\pm} = -\frac{1}{2\, m(p)}\frac{\mathrm{d}^2}{\mathrm{d}p^2} + p \pm \frac{1}{2\sqrt{p}} \tag{3.14}$$

with a momentum-dependent mass function

$$2\, m(p) = \left[\pm\frac{3}{4}\,\lambda\, p^{-\frac{1}{2}} - \frac{3}{4^{\frac{2}{3}}}(\alpha+\lambda)\right]^{-1}\ . \tag{3.15}$$

The mass function is guided by the choice of the underlying parameters α and λ. To look into the qualitative effects of the above quantum Hamiltonian one can take resort to a perturbative treatment in a similar manner as carried out in [4].

4 Summary

To summarize, we looked at a model of nonlinear oscillator describing quasi-harmonic oscillations and showed the Hamiltonian suited for it has a non-conventional double-valued structure due to the presence of a velocity-dependent potential. For small values of the coupling parameter we found that a quantum mechanical interpretation can be given to them suitable for a momentum dependent effective mass system.

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