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ON A CLASS OF HIGHER ORDER EQUATIONS WITH HIGHER GRADIENT CONVOLUTION NONLINEAR TERM

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Abstract. This paper is devoted to studying the decay rates of solutions to the nonlinear parabolic equations

$$u_t + (-1)^m \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} D^{\alpha+\beta} u = \nabla^{\mu}.\big(u\big(\varphi(\nabla^{\nu}\mathcal{N}_{t+1}) * u\big)\big), \tag{0.1}$$

where $u(0) = u_0 \in L^1(\mathbb{R}^n)$, $n > 2m$, $\mu \in \mathbb{N}^*$ and $\nu \in \mathbb{N}$ with $\max(\mu,\nu) \leq m-1$. The symbol $*$ stands for the convolution operator in the space variable and the higher order nabla differential operator ∇^{θ} denotes the vector $\big(D^{\gamma}\big)_{|\gamma|=\theta}$ with $\gamma = (\gamma_1, \cdots, \gamma_n) \in \mathbb{N}^n$. The vectorial function φ represents a nonlinearity term such that $|\varphi(X)| \leq C|X|^M$ for some real $M > (2m-\mu)/(n+\nu)$ and $\mathcal{N}_t \colon (x,t) \longmapsto \mathcal{N}(x,t)$ stands for the heat kernel related to the homogeneous operator with positive constant coefficients $a_{\alpha\beta}$.

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1 Preliminaries

The aim of this paper is to study the existence, the L^p-decay and the large time behaviour of solutions to the problem (0.1), where the real function $u\colon (x,t) \longmapsto u(x,t)$ is defined on $\mathbb{R}^n \times (0,\infty)$. The nonlinear function $\varphi\colon \mathbb{R}^\tau \longrightarrow \mathbb{R}^\eta$, where $\tau = \binom{\nu+n-1}{\nu} = \frac{(\nu+n-1)!}{\nu!(n-1)!}$, $\eta = \binom{\mu+n-1}{\mu} = \frac{(\mu+n-1)!}{\mu!(n-1)!}$ and $\max(\mu,\nu) \leq m-1$, is such that $|\varphi(X)| \leq C|X|^M$ for a non-negative real M satisfying $M > \frac{2m-\mu}{n+\nu}$. The function $\mathcal{N}_t\colon (x,t) \longmapsto \mathcal{N}(x,t)$ stands for the distributional kernel of the semigroup e^{-tL_0} associated with the homogeneous operator $L_0 = (-1)^m \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} D^{\alpha+\beta}$ of order $2m$ ($m \geq 1$), where the constant coefficients $a_{\alpha\beta}$ are assumed to be real.

We mention that the following notations will be used throughout this paper. All the computations will be done on $\mathbb{R}^n \times (0,\infty)$ with $n > 2m$. The differential operator D^γ stands for $D^\gamma_x = \frac{\partial^{\gamma_1}}{\partial x_1^{\gamma_1}} \frac{\partial^{\gamma_2}}{\partial x_2^{\gamma_2}} \cdots \frac{\partial^{\gamma_n}}{\partial x_n^{\gamma_n}}$ with $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ and the inner product $\nabla^\theta.\vec{F}$ denotes the higher divergence of the vectorial field $\vec{F}$, where $\nabla^\theta = (D^\gamma)_{|\gamma|=\theta}$ stands for the higher nabla differential operator.

We recall that in [2] we studied the class of elliptic operators L_0, where the coefficients $a_{\alpha\beta}$ were complex and satisfied the weak ellipticity condition: there exists $\delta > 0$ such that $\mathrm{Re} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}\xi^\beta\xi^\alpha \geq \delta|\xi|^{2m}$ for all $\xi \in \mathbb{R}^n$, which is equivalent to Gårding's inequality (see [1, 11]). In particular, it was shown that there exists a constant $C > 0$ such that for all $t > 0$ the heat kernel $\mathcal{N}_t$ satisfies the following Gaussian decay

$$|D^\gamma \mathcal{N}_t(x)| \leq C\, t^{-(n+|\gamma|)/2m}\, e^{-\alpha\left(\frac{|x|^{2m}}{t}\right)^{1/(2m-1)}},$$

valid for some $\alpha > 0$ and for all multi-indices $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n) \in \mathbb{N}^n$ such that $|\gamma| = \gamma_1 + \gamma_2 + \cdots + \gamma_n \leq m-1$. This corresponds to the following L^p-estimate for the higher order derivatives

$$||D^\gamma \mathcal{N}_t||_p \leq C\, t^{-\frac{n}{2m}(1-1/p)-|\gamma|/2m} \tag{1.1}$$

for all $p \in [1,\infty]$ and all $|\gamma| \leq m-1$. Using the estimate (1.1), it was shown in [14] that the fundamental solution of the heat equation satisfies

$$\left\| (D^\gamma \mathcal{N}(t) * u_0) - \left(\int_{\mathbb{R}^n} u_0 \, \mathrm{d}x \right) D^\gamma \mathcal{N}(t) \right\|_p \leq C\, t^{-\frac{n}{2m}(1-\frac{1}{p})-\frac{|\gamma|+1}{2m}}. \tag{1.2}$$

Our work is motivated by the well-known second order diffusive aggregation equations of the form $u_t = \Delta u - \nabla.(u(\nabla W * u))$ modelling the Brownian diffusion of particles interacting with a pairwise potential W (see for example [4, 5, 6, 7, 9, 13, 15] and the references therein). The class of equations studied in this work could be considered, in some sense, as generalized aggregation equations of higher order, where the interaction potential is replaced by a nonlinear term with suitable power growth and depending on the heat kernel associated to the linear part.

Before stating our main results, let us dwell on some recent literature about this kind of equations. In [13], Karch and Suzuki studied the impact of singularities of the term ∇W on the solutions. They also presented the conditions ensuring the local existence, the global existence, and the blow-up in finite time of solutions to the problem. In [7], Cañizo, Carrillo and Schonbek established the local-in-time existence of mild L^p-solutions. Precisely, assuming that $1 \leq p, q \leq \infty$ (with $p+q = pq$), $k \geq 0$, u_0 in $W^{k,p}(\mathbb{R}^n)$ and the interaction potential W is such that $\nabla W \in (L^q)^n$, they proved the existence of maximal time $0 < T \leq \infty$ and the blow-up at T (with $T < \infty$) of the unique solution

$u \in \mathcal{C}\big([0,T), W^{k,p}(\mathbb{R}^n)\big)$ to the equation. We recall that for $1 \le p \le \infty$ and $k \in \mathbb{N}$ the Sobolev space $W^{k,p}(\mathbb{R}^n) = \{u \in L^p(\mathbb{R}^n) : D^\gamma u \in L^p(\mathbb{R}^n)$ for all $\gamma \in \mathbb{N}^n$ such that $|\gamma| \le k\}$ is a Banach space equipped with the norm $||.||_{k,p} = \big(\sum_{|\gamma|\le k} ||D^\gamma.||_p^p\big)^{1/p}$.

We make use of the following fixed point theorem for bilinear forms (see [8]) to show the existence and the uniqueness of solutions to the problem at hand.

Lemma 1 *Let $\big(E, \|.\|_E\big)$ be a Banach space and let $\mathcal{F}\colon E \times E \longrightarrow E$ be a bilinear form such that $\|\mathcal{F}(u_1,u_2)\|_E \le \beta\|u_1\|_E\|u_2\|_E$ for any $(u_1,u_2) \in E \times E$. Then, for any $v \in E$ such that $\|v\|_E < \frac{1}{4}\beta$, the equation $u = v + \mathcal{F}(u,u)$ admits a unique solution u in E satisfying $\|u\|_E < 2\|v\|_E$.*

In the same paper [8], the author proved the existence of a unique global mild solution u in $\mathcal{C}\big([0,\infty), L^p\big)$ (respectively, $\mathcal{C}\big([0,\infty), W^{k,2}\big)$) under the hypotheses $u_0 \ge 0$, $u_0 \in L^1 \cap L^p$ and $\nabla W \in \big(L^q \cap L^\infty\big)^n$ (respectively, $u_0 \in W^{k,2} \cap L^1 \cap L^\infty$ for some $k \ge 1$, and $\nabla W \in \big(L^1 \cap L^\infty\big)^n$). This global solution satisfies the following L^p-bounds:

$$||u(t)||_p \le C(1+t)^{-\frac{n}{2}(1-1/p)},$$

when $n < 2k$, $p \ge 2$, $u_0 \in W^{k+1,2} \cap L^1 \cap L^\infty$, and the $W^{k,2}$-decay

$$||D^\gamma u(t)||_2 \le C(1+t)^{-(n+2m)/4},$$

when $k \ge 1$, $|\gamma| = k$ and $u_0 \in W^{k,2} \cap L^1 \cap L^\infty$. The initial data u_0 is considered to be non-negative and the interaction potential W is such that $\nabla W \in \big(L^1 \cap L^\infty\big)^n$. Also, the following asymptotic behaviour towards the heat kernel G,

$$\big\|u(t) - ||u_0||_1 G(t)\big\|_1 \le C \begin{cases} \log(t)/\sqrt{t}, & \text{if } n = 1, \\ 1/\sqrt{t}, & \text{if } n \ge 2, \end{cases}$$

for all $t \ge 1$, was established for the global solution.

Now, let us turn to the goal of this paper. The first part of our work is concerned with the local-in-time existence and uniqueness of the mild solution to the initial value problem (0.1). More precisely, under the condition $u_0 \in L^p(\mathbb{R}^n)$, we show that the unique solution is in $\mathcal{C}\big((0,\infty), W^{k,p}(\mathbb{R}^n)\big)$ for all $0 \le k \le m-1$. The proof of this result is an adaptation of the argument used, for example, in [7] or [13], which essentially relies on Lemma 1: consider the equation

$$u(t) = \mathcal{N}_t * u_0 + \underbrace{\int_0^t e^{-(t-s)L_0}\big(\nabla^\mu.\big(u\big(\varphi(\nabla^\nu \mathcal{N}_{s+1}) * u\big)\big)\big)(s)\,\mathrm{d}s}_{\mathcal{F}(u,u)} \tag{1.3}$$

in the Banach space of bounded continuous functions $\mathcal{BC}\big([0,T),\, W^{k,p}(\mathbb{R}^n)\big)$ and prove that the bilinear form $\mathcal{F}$ satisfies the assumptions of Lemma 1. To achieve this goal, we need the following useful generalized Hölder's inequality, whose proof relies on the Leibniz formula,

$$\|D^\gamma(fg)\|_r \le C_k\|f\|_{k,p}\|g\|_{k,q}, \tag{1.4}$$

valid for all $|\gamma| \le k$ and all $f \in W^{k,p}(\mathbb{R}^n)$, $g \in W^{k,q}(\mathbb{R}^n)$ with $1/p + 1/q = 1/r$, and the generalized Young's inequality

$$||f * g||_{k,r} \leq ||f||_q ||g||_{k,p}, \tag{1.5}$$

valid for all $f \in L^q(\mathbb{R}^n)$, $g \in W^{k,p}(\mathbb{R}^n)$ with $1 \leq p, q, r \leq \infty$ and $1/p + 1/q = 1 + 1/r$. Also, a classical argument based on integration with respect to x and the Fourier transform allows to show that the unique solution to (0.1) satisfies for all $t \geq 0$ the conservation of mass $\int_{\mathbb{R}^n} u(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^n} u_0(x)\,\mathrm{d}x$ and the contraction property

$$||u(t)||_1 \leq ||u_0||_1. \tag{1.6}$$

The second part of our work is devoted to proving the L^p-decay of the solution to (0.1) and its higher order space derivatives by using the Duhamel formula. In precise terms, by combining the Gaussian upper bounds for the heat kernel with the growth condition of the nonlinear term, we prove (for some range of p and under some conditions on the parameters μ, ν) that the following L^p-bounds hold

$$\begin{cases} \|u(t)\|_p \leq C\|u_0\|_1\, t^{-\frac{n}{2m}(1-\frac{1}{p})} \\ \|D^\gamma u(t)\|_p \leq C\, t^{-\frac{n}{2m}(1-\frac{1}{p})-\frac{|\gamma|}{2m}} \end{cases}$$

for all $\gamma \in \mathbb{N}^n$ such that $\mu + |\gamma| \leq m - 1$. The approach used to show these results relies on some technical tools, including a classical Gronwall's lemma stated as follows.

Lemma 2 (Gronwall) *Let $A > 0$ and let f and g be two positive continuous functions on $[0,T]$ such that $f(t) \leq A + \int_0^t g(s)f(s)\,\mathrm{d}s$ for all $t \in [0,T]$. Then, $f(t) \leq A\exp\big(\int_0^t g(s)\,\mathrm{d}s\big)$ for all $t \in [0,T]$.*

Lastly, we investigate the large time behaviour of the solution to the problem (0.1). We prove that the asymptotic behaviour is dictated by the heat kernel as follows

$$\|D^\gamma u(t) - M_0\, D^\gamma \mathcal{N}(t)\|_p \leq C\, t^{-\frac{n}{2m}(1-\frac{1}{p})-\frac{|\gamma|+1}{2m}}.$$

This result is proved for u_0 in the weighted space $L^1\big(\mathbb{R}^n,\, 1+|x|\big) = \{f \in L^1(\mathbb{R}^n) : \int_{\mathbb{R}^n}(1+|x|)|f(x)|\,\mathrm{d}x < \infty\}$ which is dense in $L^1(\mathbb{R}^n)$. It is worth mentioning that the following well-known technical lemma (see for example [3]) is needed to handle the second term on the right-hand side of (1.3).

Lemma 3 *Let a, b and c be real numbers such that $a < 1$, $b > 0$, $c < 1$. Then, there exists a constant $C > 0$ such that*

$$\int_0^t (t-s)^{-a}\,(1+s)^{-b}\,s^{-c}\,\mathrm{d}s \leq C \begin{cases} t^{-a}, & \text{if } b+c > 1, & \text{(i)} \\ t^{-a}\,\ln(1+t), & \text{if } b+c = 1, & \text{(ii)} \\ t^{1-a-c}\,(1+t)^{-b}, & \text{if } b+c < 1. & \text{(iii)} \end{cases}$$

2 Short time existence

Before stating the main result, let us give the definition of a solution to the Cauchy problem (0.1) adopted in this paper.

Definition 1 *Let $T \in (0,+\infty]$ and $u_0 \in L^p(\mathbb{R}^n)$ with $1 \le p \le \infty$. By a solution u to the equation* (0.1) *we mean a mild L^p-solution on $[0,T)$,* i.e., $u \in L^1_{loc}\big([0,T),\, L^p(\mathbb{R}^n)\big)$ *satisfies the Duhamel integral formula*

$$u(t) = e^{-tL_0}u_0 + \int_0^t e^{-(t-s)L_0}\big(\nabla^\mu.\big(u\big(\varphi(\nabla^\nu \mathcal{N}_{s+1}) * u\big)\big)\big)(s)\,\mathrm{d}s \tag{2.1}$$

for all $t \in (0,T)$.

Theorem 1 *Let $p \in [1,\infty]$ and $u_0 \in W^{k,p}(\mathbb{R}^n)$ with $0 \le k \le m-1$. Then, there exists a unique solution $u \in \mathcal{C}\big([0,T), W^{k,p}(\mathbb{R}^n)\big)$ to the problem* (0.1).

Proof. Let $T > 0$ and $0 \le k \le m-1$. On the Banach space $E := \mathcal{BC}\big([0,T),\, W^{k,p}(\mathbb{R}^n)\big)$ endowed with the norm $|||w|||_{k,p} = \sup_{t\in[0,T)} \|w(t)\|_{k,p}$, we define the following bilinear form

$$\mathcal{F}(u,\psi) = (-1)^\mu \int_0^t \Big(\nabla^\mu \mathcal{N}(t-s) * \Big(u(s)\big(\varphi(\nabla^\nu \mathcal{N}(s+1)) * \psi(s)\big)\Big)\Big)\,\mathrm{d}s$$

for $t \in [0,T)$.

For all $t \in [0,T)$ and all multi-indices $\gamma \in \mathbb{N}^n$ such that $|\gamma| \le k$, by Young's inequality, thanks to (1.1) and Hölder's property (1.4), we have

$$\begin{aligned}
\big\|D^\gamma \mathcal{F}(u,\psi)\big\|_p &\le \int_0^t \Big\|\nabla^\mu \mathcal{N}(t-s) * \Big(D^\gamma\Big(u(s)\big(\varphi(\nabla^\nu \mathcal{N}(s+1)) * \psi(s)\big)\Big)\Big)\Big\|_p \,\mathrm{d}s \\
&\le \int_0^t \big\|\nabla^\mu \mathcal{N}(t-s)\big\|_1 \Big\|D^\gamma\Big(u(s)\big(\varphi(\nabla^\nu \mathcal{N}(s+1)) * \psi(s)\big)\Big)\Big\|_p \,\mathrm{d}s \\
&\le C\int_0^t (t-s)^{-\frac{\mu}{2m}}\, \|u(s)\|_{k,p}\, \underbrace{\|\varphi(\nabla^\nu \mathcal{N}(s+1)) * \psi(s)\|_{k,\infty}}_{\xi(s)}\,\mathrm{d}s.
\end{aligned}$$

Also, by Young's inequality (1.5), for all q such that $p+q = pq$ we obtain

$$\begin{aligned}
\xi(s) &\le \|\varphi(\nabla^\nu \mathcal{N}(s+1))\|_q\, \|\psi(s)\|_{k,p} \\
&\le C\,\big\|\nabla^\nu \mathcal{N}(s+1)\big\|_{qM}^M\, \|\psi(s)\|_{k,p} \\
&\le C\,(s+1)^{-\frac{1}{2mq}(q(n+\nu)M-n)}\, \|\psi(s)\|_{k,p}.
\end{aligned}$$

It follows from these estimates that

$$\begin{aligned}
\big\|D^\gamma \mathcal{F}(u,\psi)\big\|_p &\le C\int_0^t (t-s)^{-\frac{\mu}{2m}}\,(s+1)^{-\frac{1}{2mq}(q(n+\nu)M-n)}\,\|u(s)\|_{k,p}\,\|\psi(s)\|_{k,p}\,\mathrm{d}s \\
&\le C\int_0^t (t-s)^{-\frac{\mu}{2m}}\,(s+1)^{-(1-\frac{n}{2mq}-\frac{\mu}{2m})}\,\|u(s)\|_{k,p}\,\|\psi(s)\|_{k,p}\,\mathrm{d}s.
\end{aligned}$$

Then, by taking the supremum for $t \in [0,T)$, we obtain

$$|||D^\gamma \mathcal{F}(u,\psi)|||_{k,p} \le C\,T^{n/2mq}\,|||u(s)|||_{k,p}\,|||\psi(s)|||_{k,p}.$$

To conclude the proof it remains to use Lemma 1 with $\beta = C\,T^{n/2mq}$. $\square$

3 L^p-decay of solutions

The following result is concerned with the L^p-decay of the solution to the initial value problem (0.1).

Theorem 2 *For all p such that $\frac{n}{n-\mu} < p < \frac{n}{n-2m}$, there exists a constant $C > 0$ such that for all $t > 0$ the solution u of (0.1) satisfies*

$$\|u(t)\|_p \leq C\|u_0\|_1\, t^{-\frac{n}{2m}(1-\frac{1}{p})}.$$

Proof. From the Duhamel formula (2.1), we derive the following estimate

$$\|u(t)\|_p \leq \|\mathcal{N}_t * u_0\|_p + \underbrace{\int_0^t \big\|\nabla^\mu \mathcal{N}(t-s) * \big(u(s)\big(\varphi(\nabla^\nu \mathcal{N}(s+1)) * u(s)\big)\big)\big\|_p \, \mathrm{d}s}_{\mathcal{X}(t)}.$$

According to the contraction property (1.6), the L^p-estimates on the kernel (1.1), Young's and Hölder's inequalities, we obtain

$$\|\mathcal{N}_t * u_0\|_p \leq \|\mathcal{N}_t\|_p \|u_0\|_1 \leq C\|u_0\|_1\, t^{-\frac{n}{2m}(1-\frac{1}{p})}$$

and

$$\begin{aligned}
\mathcal{X}(t) &\leq \int_0^t \big\|\nabla^\mu \mathcal{N}(t-s)\big\|_1 \big\|u(s)\big(\varphi(\nabla^\nu \mathcal{N}_{s+1}) * u\big)(s)\big\|_p \, \mathrm{d}s \\
&\leq C \int_0^t (t-s)^{-\frac{\mu}{2m}} \big\|\big(\varphi(\nabla^\nu \mathcal{N}_{s+1}) * u\big)(s)\big\|_\infty \|u(s)\|_p \, \mathrm{d}s \\
&\leq C \int_0^t (t-s)^{-\frac{\mu}{2m}} \|\varphi(\nabla^\nu \mathcal{N}(s+1))\|_\infty \|u(s)\|_1 \|u(s)\|_p \, \mathrm{d}s \\
&\leq C \int_0^t (t-s)^{-\frac{\mu}{2m}} \|\nabla^\nu \mathcal{N}(s+1)\|_\infty^M \|u(s)\|_1 \, \|u(s)\|_p \, \mathrm{d}s \\
&\leq C\|u_0\|_1 \int_0^t (t-s)^{-\frac{\mu}{2m}} (s+1)^{-\frac{(n+\nu)M}{2m}} \|u(s)\|_p \, \mathrm{d}s,
\end{aligned}$$

where C is a generic constant which may change its value in the sequel. These estimates yield

$$\|u(t)\|_p \leq C\|u_0\|_1 \left(t^{-\frac{n}{2m}(1-\frac{1}{p})} + \int_0^t (t-s)^{-\frac{\mu}{2m}} (s+1)^{-\frac{(n+\nu)M}{2m}} \|u(s)\|_p \, \mathrm{d}s\right),$$

whereupon

$$h(t) \leq C_0 + C_0\, t^\kappa \underbrace{\int_0^t (t-s)^{-\frac{\mu}{2m}} (s+1)^{-\tau} s^{-\kappa} h(s) \, \mathrm{d}s}_{\mathcal{Z}(t)}, \tag{3.1}$$

where

$$\tau = \frac{(n+\nu)M}{2m}, \quad \kappa = \frac{n}{2m}(1-1/p), \quad h(t) = t^\kappa \|u(t)\|_p \quad \text{and} \quad C_0 = C\|u_0\|_1.$$

Write $\mathcal{Z}(t) = \int_0^{(1-\varepsilon)t} \cdots + \int_{(1-\varepsilon)t}^{t} \cdots := \mathcal{Z}_1(t) + \mathcal{Z}_2(t)$ for $\varepsilon > 0$ small enough. Observe that

$$\mathcal{Z}_1(t) \le (\varepsilon t)^{-\mu/2m} \int_0^{(1-\varepsilon)t} (s+1)^{-\tau}\, s^{-\kappa} h(s)\,\mathrm{d}s$$

and for $f(t) = \sup_{s\in[0,t]} h(s)$, we obtain

$$\mathcal{Z}_2(t) \le \big((1-\varepsilon)t\big)^{-\kappa} f(t) \int_{(1-\varepsilon)t}^{t} (t-s)^{-\frac{\mu}{2m}}\,\mathrm{d}s \le \frac{(1-\varepsilon)^{-\kappa}(\varepsilon T)^{1-\mu/2m}}{1-\mu/2m} f(t)\, t^{-\kappa}.$$

It then follows from (3.1) that for all $t \in (0,T]$ we have

$$h(t) \le C_0 + C_1\, f(t) + C_0\, \varepsilon^{-\mu/2m}\, t^{\kappa-\mu/2m} \int_0^{(1-\varepsilon)t} (s+1)^{-\tau}\, s^{-\kappa} h(s)\,\mathrm{d}s, \tag{3.2}$$

where $C_1 = C_0\, \frac{(1-\varepsilon)^{-\kappa}(\varepsilon T)^{1-\mu/2m}}{1-\mu/2m}$. Notice that the conditions given on the parameters guarantee that $\frac{\mu}{2m} < \kappa < 1$. Now, by taking ε small enough to get $0 < C_1 < \epsilon_0 < 1$ and applying the supremum for $t \in (0,\mathcal{T}]$ on the right- and left-hand side of (3.2), we find the following inequality

$$f(\mathcal{T}) \le C_0 + \epsilon_0\, f(\mathcal{T}) + C_0\, \varepsilon^{-\mu/2m}\, \mathcal{T}^{\kappa-\mu/2m} \int_0^{(1-\varepsilon)\mathcal{T}} (s+1)^{-\tau}\, s^{-\kappa} f(s)\,\mathrm{d}s,$$

which for all $\mathcal{T} \in [0,T]$ implies that

$$f(\mathcal{T}) \le \frac{C_0}{1-\epsilon_0} + \frac{C_0\, \varepsilon^{-\mu/2m}\, \mathcal{T}^{\kappa-\mu/2m}}{1-\epsilon_0} \int_0^{\mathcal{T}} (s+1)^{-\tau}\, s^{-\kappa} f(s)\,\mathrm{d}s. \tag{3.3}$$

Applying weakly singular Gronwall's lemma (see Lemma 2) to (3.3), yields

$$f(\mathcal{T}) \le A\, \exp\!\bigg(B \int_0^{\mathcal{T}} (s+1)^{-\tau}\, s^{-\kappa}\,\mathrm{d}s\bigg)$$

with $A = \frac{C_0}{1-\epsilon_0}$ and $B = \frac{C_0\, \varepsilon^{-\mu/2m}\, \mathcal{T}^{\kappa-\mu/2m}}{1-\epsilon_0}$. Consequently, for all $t \in [0,T]$,

$$\|u(t)\|_p \le C\, t^{-\frac{n}{2m}(1-\frac{1}{p})},$$

where $C = A\, \exp\big(B \int_0^T (s+1)^{-\tau}\, s^{-\kappa}\,\mathrm{d}s\big)$. □

Comments 1 (1) *For the constant C we have the following estimates*

$$C \le A \begin{cases} e^{cB}, & \textit{if } \tau+\kappa > 1, \\ (1+T)^{cB}, & \textit{if } \tau+\kappa = 1, \\ \exp\big(cBT^{1-\kappa}(1+T)^{-\tau}\big), & \textit{if } \tau+\kappa < 1. \end{cases}$$

(2) *The second part of the proof of Theorem* 2 *allows to state the following technical result* (*non-classical Gronwall's lemma*)*: Let $K > 0$, $\delta > 0$, $0 \le \lambda < \theta < 1$ and let f be a positive continuous function on $[0,T]$ such that for all $t \in (0,T]$,*

$$f(t) \le K\bigg(t^{-\theta} + \int_0^t (t-s)^{-\lambda}(s+1)^{-\delta} f(s)\,\mathrm{d}s\bigg).$$

Then, there exists a constant $C > 0$ such that for all $t \in (0,T]$ we have

$$f(t) \le C\, K\, t^{-\theta}.$$

The arguments used are an adaptation of the ones employed in, for example, [10, 12].

(3) *If $u_0 \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, then we obtain the following L^p-decay*:

$$\|u(t)\|_p \le C\,\|u_0\|_p \exp\big(c\|u_0\|_1\, t^{1-\mu/2m}\big). \tag{3.4}$$

Indeed, as before, $\|\mathcal{N}_t * u_0\|_p \le \|\mathcal{N}_t\|_1 \|u_0\|_p \le C\|u_0\|_p$ and

$$\begin{aligned}\|u(t)\|_p &\le C\|u_0\|_p + C\|u_0\|_1 \int_0^t (t-s)^{-\frac{\mu}{2m}}\,(s+1)^{-\frac{(n+\nu)M}{2m}}\,\|u(s)\|_p\,\mathrm{d}s \\ &\le C\|u_0\|_p + C\|u_0\|_1 \int_0^t (t-s)^{-\frac{\mu}{2m}}\,\|u(s)\|_p\,\mathrm{d}s,\end{aligned}$$

which implies (3.4) *according to the following Gronwall's lemma* (*see, for example,* [7, 12]): *Let A, B and λ be real numbers such that $A > 0$, $B > 0$, $0 < \lambda < 1$ and let $f\colon [0,T] \to [0,\infty)$* (*where $T \in [0,\infty]$*) *be a continuous function satisfying $f(t) \le A + B\int_0^t (t-s)^{-\lambda} f(s)\,\mathrm{d}s$ for all $t \in [0,T)$. Then, there exists a constant $C > 0$ such that for all $t \in [0,T)$, we have $f(t) \le A\exp(B\,t^{1-\lambda})$.*

4 Large time behaviour

The main result of this section deals with the asymptotic behaviour of solutions to (0.1). We show that the higher derivatives of solutions behave like the heat kernel in the following sense.

Theorem 3 *Let $\gamma \in \mathbb{N}^n$ be such that $\mu + |\gamma| \le m-1$. Then, for all $\frac{n}{n-\mu} < p < \frac{n}{n-2m}$ there exists a constant $C > 0$ such that for all $t > 0$ we have*

$$\|D^\gamma u(t)\|_p \le C\,t^{-\frac{n}{2m}(1-\frac{1}{p})-\frac{|\gamma|}{2m}} \tag{4.1}$$

and

$$\|D^\gamma u(t) - M_0\,D^\gamma \mathcal{N}(t)\|_p \le C\,t^{-\frac{n}{2m}(1-\frac{1}{p})-\frac{|\gamma|}{2m}}\,\mathcal{P}(t), \tag{4.2}$$

where $M_0 = \int_{\mathbb{R}^n} u_0(x)\,\mathrm{d}x$ and

$$\mathcal{P}(t) = \begin{cases} t^{-1/2m}, & \text{if } M > \frac{2m}{n+\nu}, \\ t^{-1/2m}\ln(1+t), & \text{if } M = \frac{2m}{n+\nu}, \\ \max\Big(t^{-1/2m},\, t^{-\frac{\mu+(n+\nu)M-2m}{2m}}\Big), & \text{if } \frac{2m-\mu}{n+\nu} < M < \frac{2m}{n+\nu}. \end{cases}$$

Proof. We begin by giving the upper bounds for higher order derivatives. According to (2.1), for all multi-indices $\gamma \in \mathbb{N}^n$ such that $\mu + |\gamma| \le m-1$ we have

$$D^\gamma u(t) = (D^\gamma \mathcal{N}_t * u_0) + \int_0^t \Big(D^\gamma \mathcal{N}(t-s) * \nabla^\mu.\big(u(\varphi(\nabla^\nu \mathcal{N}_{s+1}) * u)\big)\Big)(s)\,\mathrm{d}s.$$

Using (1.1), we obtain

$$\begin{aligned}\|D^\gamma u(t)\|_p &\le \|D^\gamma \mathcal{N}_t * u_0\|_p + \mathcal{A}(t) \\ &\le \|D^\gamma \mathcal{N}_t\|_p \|u_0\|_1 + \mathcal{A}(t) \\ &\le C\|u_0\|_1\, t^{-\frac{n}{2m}(1-1/p)-|\gamma|/2m} + \mathcal{A}(t),\end{aligned}$$

where

$$\mathcal{A}(t) = \int_0^t \big\| \nabla^\mu \big(D^\gamma \mathcal{N}(t-s) \big) * \big(u(s) \big(\varphi(\nabla^\nu \mathcal{N}(s+1)) * u(s) \big) \big) \big\|_p \, \mathrm{d}s. \tag{4.3}$$

One the other hand, the computations on $\mathcal{A}(t)$ yield

$$\begin{aligned}
\mathcal{A}(t) &\le \int_0^t \|\nabla^\mu \big(D^\gamma \mathcal{N}(t-s) \big)\|_p \, \|u(s) \big(\varphi(\nabla^\nu \mathcal{N}(s+1)) * u(s) \big)\|_1 \, \mathrm{d}s \\
&\le \int_0^t \|\nabla^\mu \big(D^\gamma \mathcal{N}(t-s) \big)\|_p \, \|u(s)\|_p \, \|\varphi(\nabla^\nu \mathcal{N}(s+1))\|_q \, \|u(s)\|_1 \, \mathrm{d}s \\
&\qquad\qquad\qquad (\text{where } 1/p + 1/q = 1) \\
&\le \|u_0\|_1 \int_0^t \|\nabla^\mu \big(D^\gamma \mathcal{N}(t-s) \big)\|_p \, \|u(s)\|_p \, \|\nabla^\nu \mathcal{N}(s+1)\|_{qM}^M \, \mathrm{d}s \\
&\le C \, \|u_0\|_1 \int_0^t (t-s)^{-\frac{n}{2m}(1-\frac{1}{p}) - \frac{\mu+|\gamma|}{2m}} \, s^{-\frac{n}{2m}(1-\frac{1}{p})} \, (1+s)^{-(1-\frac{\mu}{2m}-\frac{n}{2mq})} \, \mathrm{d}s \\
&\le C \, \|u_0\|_1 \, t^{1-\frac{n}{mq}-\frac{\mu+|\gamma|}{2m}} \, (1+t)^{-(1-\frac{\mu}{2m}-\frac{n}{2mq})} \le C t^{-\frac{n}{2m}(1-1/p) - |\gamma|/2m}
\end{aligned}$$

thanks to (iii) of Lemma 3.

To prove (4.2), we use an approach similar to the above one and a density argument. Suppose that $u_0 \in L^1\big(\mathbb{R}^n, \, 1+|x|\big)$ and let the multi-index γ be such that $\mu + |\gamma| \le m-1$. Then, we have

$$\begin{aligned}
D^\gamma u(t) - M_0 \, D^\gamma \mathcal{N}(t) = \big((D^\gamma \mathcal{N}(t) * u_0) - M_0 \, D^\gamma \mathcal{N}(t) \big) + \\
+ \int_0^t \Big(\nabla^\mu \big(D^\gamma \mathcal{N}(t-s) \big) * \Big(u(s) \big(\varphi(\nabla^\nu \mathcal{N}(s+1)) * u(s) \big) \Big) \Big) \, \mathrm{d}s;
\end{aligned}$$

so

$$\big\| D^\gamma u(t) - M_0 \, D^\gamma \mathcal{N}(t) \big\|_p \le \mathcal{A}(t) + \mathcal{B}(t),$$

where $\mathcal{B}(t) := \big\| (D^\gamma \mathcal{N}(t) * u_0) - M_0 \, D^\gamma \mathcal{N}(t) \big\|_p$ and $\mathcal{A}(t)$ is given by (4.3).

First, for the fundamental solution of the heat equation, (1.2) implies

$$\mathcal{B}(t) \le C \, t^{-\frac{n}{2m}(1-\frac{1}{p}) - \frac{|\gamma|}{2m} - \frac{1}{2m}}.$$

As before, the estimates on the term $\mathcal{A}(t)$ yield

$$\begin{aligned}
\mathcal{A}(t) &\le C \, \|u_0\|_1 \int_0^t (t-s)^{-(\frac{n}{2mq} + \frac{\mu+|\gamma|}{2m})} (1+s)^{-(\frac{(n+\nu)M}{2m} - \frac{n}{2mq})} \, s^{-\frac{n}{2mq}} \, \mathrm{d}s \\
&\le C \, \|u_0\|_1 t^{-\frac{n}{2m}(1-\frac{1}{p}) - \frac{|\gamma|}{2m}} \begin{cases} t^{-\mu/2m}, & \text{if } M > \frac{2m}{n+\nu}, \\ t^{-\mu/2m} \ln(1+t), & \text{if } M = \frac{2m}{n+\nu}, \\ t^{1-\frac{\mu+(n+\nu)M}{2m}}, & \text{if } \frac{2m-\mu}{n+\nu} < M < \frac{2m}{n+\nu}, \end{cases}
\end{aligned}$$

thanks to Lemma 3. These bounds imply (4.2) for $u_0 \in L^1\big(\mathbb{R}^n, \, 1+|x|\big)$ and Theorem 3 is then completely proved by using the density of $L^1\big(\mathbb{R}^n, \, 1+|x|\big)$ in $L^1(\mathbb{R}^n)$. □

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