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IMPULSIVE STOCHASTIC FRACTIONAL ORDER INTEGRO-DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

Mohd. Nadeem*, **Jaydev Dabas**†
Department of Applied Science and Engineering, Indian Institute of Technology Roorkee, Saharanpur Campus, Saharanpur, India, Pin-247001

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Abstract. In this paper, we prove the existence of mild solutions for a class of fractional order impulsive stochastic integro-differential equations with infinite delay. The existence results are proved by using the fixed point techniques in a Hilbert space. We also provide an example to illustrate the existence results.

Keywords: Existence and uniqueness, fractional order differential equations, impulsive conditions, stochastic differential equations.

2010 Mathematics Subject Classification: 26A33, 34A12, 34A37, 34K50.

1 Introduction

During the last two decades fractional differential equations have received a great deal of interest due to their numerous applications in various fields of engineering and science such as viscoelasticity, porous media, electrochemistry, control theory, physics and so on. Fractional differential equations are also used to describe the memory and hereditary properties of various materials and processes. For significant developments in fractional differential equations one can see the monographs of Kilbas *et al.* [16], Podlubny [24] and the papers [9, 10, 11, 18, 20] and references therein.

On the other hand, stochastic differential equations (SDEs) have been widely applied to model problems in various fields such as engineering, physics, biology, finance and economics, etc. in

*e-mail address: mohdnadeem.jmi@gmail.com
†e-mail address: jay.dabas@gmail.com

which systems fluctuate due to noise and other disturbances. In many areas of engineering and science the future state of many dynamical systems does not depend only on the present state but also on their past history. This leads to stochastic functional differential equations. SDEs have played a very important role in many fields of real life such as forecast of population growth, optimal pricing, etc.; for more details on SDEs one can see the monographs [21, 22, 25] and references therein. To include time in memory we use the integro-differential equations.

SDEs with infinite delay have become an area of emerging research as they have played an important role in the last few years to translate various problems from physical and social sciences into mathematical models. Sometimes we observe an abrupt change in a dynamical system. To model such type of problems we consider the impulsive effects. For more details one can see the cited papers [4, 5, 7, 14, 19, 23, 26, 29].

In the case of fractional order deterministic models, Chauhan and Dabas [8] considered a fractional integro-differential equation with impulsive conditions and discussed the existence of mild solutions using the single base point technique as in [12]. Gautam *et al.* [13] addressed the existence and uniqueness of mild solutions to a class of fractional impulsive integro-differential equations with state dependent delay and establihed the results by using fixed point technique and following the concept as in [12]. Recently, in the case of fractional stochastic equations, Sakthivel *et al.* [29] have considered a class of fractional order impulsive SDEs with infinite delay and have established the existence of mild solutions using the technique of multi-base point as in article [2] by means of the Banach, Krasnoselskii's and Schaefer's fixed point theorems. In [7] the authors used a technique similar to that used in [29] to define solutions and discuss the existence of mild solutions to a class of stochastic fractional impulsive neutral integro-differential equation with infinite delay.

Motivated by the above-mentioned works [7, 8, 13, 29], in this article we consider the following fractional order functional SDE in the following form:

$$ {}^cD_t^\alpha x(t) = Ax(t) + \int_0^t q(t-s)b(s,x_s)\,\mathrm{d}s + \mathbb{J}^{1-\alpha}\left[f(t,x_t) + g(t,x_t)\frac{\mathrm{d}w(t)}{\mathrm{d}t}\right], \quad t \in J = [0,T],\ t \neq t_k, \tag{1.1} $$

$$ x(t) = \phi(t),\ t \in (-\infty, 0], \tag{1.2} $$

$$ \Delta x(t_k) = I_k(x(t_k^-)),\ k = 1,2,\dots,m, \tag{1.3} $$

where ${}^cD_t^\alpha$ denotes the fractional derivative of order $\alpha \in (0,1)$ of Caputo's type, $A\colon D(A) \subset \mathbb{H} \to \mathbb{H}$ is a closed linear sectorial operator defined on a Hilbert space $(\mathbb{H}, \|\cdot\|)$, $\mathbb{J}^{1-\alpha}$ is the Riemann–Liouville fractional integral of order $1-\alpha > 0$ and $q\colon J \to \mathbb{H}$ is a continuous function. The functions b, f, g are given and satisfy some assumptions which will be specified later. Infinite delay is given in the equation (1.2). We assume that $x_t\colon (-\infty, 0] \to \mathbb{H}$, defined as $x_t(\theta) = x(t+\theta)$ for $\theta \leq 0$, belongs to an abstract phase space $\mathfrak{B}_h$. Here $0 \leq t_0 < t_1 < \cdots < t_{m+1} \leq T$, and for $k = 1,2,\dots,m$, $I_k \in C(\mathbb{H},\mathbb{H})$ are bounded functions and $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, where $x(t_k^-) = \lim_{h\to 0} x(t_k - h)$ and $x(t_k^+) = \lim_{h\to 0} x(t_k + h)$ are the left-hand and right-hand limits of $x(t)$ at $t = t_k$, respectively; also we take $x(t_k^-) = x(t_k)$.

However, to the best of our knowledge, the existence results for impulsive stochastic fractional order integro-differential equations in the form (1.1)–(1.3) is an untreated topic yet. This fact is the motivation of the present work. The existence results are shown by using the Banach and Krasnoselskii's fixed point theorems. We closely follows the approach of [29]. The paper is organized as follows. In Section 2, we recall some preliminaries and cite the appropriate references. Section

3 is devoted to stating and proving the main results. In the last section an example is presented to illustrate our approach.

2 Preliminaries

Let $\mathbb{H}$, $\mathbb{K}$ be two separable Hilbert spaces with the corresponding norm $\|\cdot\|$. The symbol $\mathcal{L}(\mathbb{K},\mathbb{H})$ stands for the space of bounded linear operators from $\mathbb{K}$ into $\mathbb{H}$. We use $(\cdot,\cdot)$ to denote the inner product of $\mathbb{H}$ and $\mathbb{K}$, since no confusion should arise. To avoid repetition, we refer the readers to the papers [27, 28, 29] for more details regarding complete filtered spaces and properties of $\mathbb{Q}$-Wiener processes. Further, for basic definitions and properties of Caputo's derivative, Riemann–Liouville fractional integral operator and Mittag-Lefller function we refer to the book [24]. The details concerning the solution and sectorial operators can be found in [1] and [15]. For the notion of the α-resolvent family one can see the paper [3].

Now, we introduce the abstract phase space $\mathfrak{B}_h$ required in this work and introduced in the papers [27, 28]. Assume that $h\colon(-\infty,0]\to(0,\infty)$ with $l=\int_{-\infty}^{0}h(t)\,\mathrm{d}t<\infty$ is a continuous function. The abstract phase space $\mathfrak{B}_h$ is defined by

$$\mathfrak{B}_h=\left\{\phi\colon(-\infty,0]\to\mathbb{H}\ :\ \begin{array}{l}(E|\phi(\theta)|^2)^{1/2}\text{ is a bounded and measurable function on }[-a,0],\\ \text{where }a>0\text{, and }\int_{-\infty}^{0}h(s)\sup_{s\le\theta\le0}(E|\phi(\theta)|^2)^{1/2}\,\mathrm{d}s<\infty\end{array}\right\}.$$

If $\mathfrak{B}_h$ is endowed with the norm

$$\|\phi\|_{\mathfrak{B}_h}=\int_{-\infty}^{0}h(s)\sup_{s\le\theta\le0}(E|\phi(\theta)|^2)^{1/2}\,\mathrm{d}s,\ \phi\in\mathfrak{B}_h,$$

then $(\mathfrak{B}_h,\|\cdot\|_{\mathfrak{B}_h})$ is a Banach space [27, 28].

Now, we consider the space

$$\mathfrak{B}'_h=\left\{x\colon(-\infty,T]\to\mathbb{H}\ :\ \begin{array}{l}x|_{J_k}\in C(J_k,\mathbb{H})\text{ and there exist }x(t_k^+)\text{ and }x(t_k^-)\\ \text{with }x(t_k)=x(t_k^-),\,x_0=\phi\in\mathfrak{B}_h,\,k=1,2,\dots,m\end{array}\right\},$$

where $x|_{J_k}$ is the restriction of x to $J_k=(t_k,t_{k+1}]$, $k=0,1,2,\dots,m$. The function $\|\cdot\|_{\mathfrak{B}'_h}$ defined by

$$\|x\|_{\mathfrak{B}'_h}=\|\phi\|_{\mathfrak{B}_h}+\sup_{s\in[0,T]}(E\|x(s)\|^2)^{1/2},\quad x\in\mathfrak{B}'_h,$$

is a semi-norm in $\mathfrak{B}'_h$.

Lemma 1 ([28]) *Suppose that $v\in\mathfrak{B}'_h$. Then, $v_t\in\mathfrak{B}_h$ for $t\in J$. Furthermore,*

$$l(E\|v(t)\|^2)^{1/2}\le l(\|v_t\|^2)^{1/2}\le l\sup_{s\in[0,t]}(E\|v(s)\|^2)^{1/2}+\|v_0\|_{\mathfrak{B}_h},$$

where $l=\int_{-\infty}^{0}h(s)\,\mathrm{d}s<\infty$.

Definition 1 *A measurable $\mathcal{F}_t$-adapted stochastic process $x\colon(-\infty,T]\to\mathbb{H}$ is called a mild solution of the system* (1.1)–(1.3) *if $x_0=\phi\in\mathfrak{B}_h$, $\Delta x|_{t=t_k}=I_k(x(t_k^-))$, $k=1,2,\dots,m$, the restriction of $x(\cdot)$ to each connected component of $[0,T)\setminus\{t_1,\dots,t_m\}$ is continuous and $x(t)$ satisfies the following integral equation*

$$x(t)=\begin{cases} S_\alpha(t)\phi(0)+\displaystyle\int_0^t T_\alpha(t-s)\Big[\int_0^s q(s-\eta)b(\eta,x_\eta)\,\mathrm{d}\eta\Big]\mathrm{d}s \\ \quad+\displaystyle\int_0^t S_\alpha(t-s)f(s,x_s)\,\mathrm{d}s+\int_0^t S_\alpha(t-s)g(s,x_s)\,\mathrm{d}w(s), \quad t\in(0,t_1], \\ S_\alpha(t)\phi(0)+S_\alpha(t-t_1)I_1(x(t_1^-)) \\ \quad+\displaystyle\int_0^t T_\alpha(t-s)\Big[\int_0^s q(s-\eta)b(\eta,x_\eta)\,\mathrm{d}\eta\Big]\mathrm{d}s \\ \quad+\displaystyle\int_0^t S_\alpha(t-s)f(s,x_s)\,\mathrm{d}s+\int_0^t S_\alpha(t-s)g(s,x_s)\,\mathrm{d}w(s), \quad t\in(t_1,t_2], \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ S_\alpha(t)\phi(0)+\displaystyle\sum_{i=1}^m S_\alpha(t-t_i)I_i(x(t_i^-)) \\ \quad+\displaystyle\int_0^t T_\alpha(t-s)\Big[\int_0^s q(s-\eta)b(\eta,x_\eta)\,\mathrm{d}\eta\Big]\mathrm{d}s \\ \quad+\displaystyle\int_0^t S_\alpha(t-s)f(s,x_s)\,\mathrm{d}s+\int_0^t S_\alpha(t-s)g(s,x_s)\,\mathrm{d}w(s), \quad t\in(t_m,T], \end{cases} \tag{2.1}$$

where

$$S_\alpha(t)=\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha I-A)^{-1}\,\mathrm{d}\lambda,$$
$$T_\alpha(t)=\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda^\alpha I-A)^{-1}\,\mathrm{d}\lambda,$$

are called analytic solutions operator and $\alpha-$resolvent family, respectively, and Γ is a suitable path lying in $\sum_{\theta,\omega}$.

To establish our desired results we consider the following assumptions.

(H_0) If $\alpha\in(0,1)$ and $A\in\mathbb{A}^\alpha(\theta_0,\omega_0)$, then for any $x\in\mathbb{H}$ and $t>0$ we have $\|S_\alpha(t)\|\le Me^{\omega t}$ and $\|T_\alpha(t)\|\le Ce^{\omega t}(1+t^{\alpha-1})$, $\omega>\omega_0$. Thus, we have

$$\|S_\alpha(t)\|\le\widetilde{M}_S \text{ and } \|T_\alpha(t)\|\le t^{\alpha-1}\widetilde{M}_T,$$

where $\widetilde{M}_S=\sup_{0\le t\le T}\|S_\alpha(t)\|$ and $\widetilde{M}_T=\sup_{0\le t\le T}Ce^{\omega t}(1+t^{1-\alpha})$ (for more details see [15]).

(H_1) The functions $f,b\colon J\times\mathfrak{B}_h\to\mathbb{H}$ are continuous and there exist non-negative constants L_f,L_b such that

$$E\|f(t,\gamma)-f(t,\psi)\|_{\mathbb{H}}^2\le L_f\|\gamma-\psi\|_{\mathfrak{B}_h}^2,$$
$$E\|b(t,\gamma)-b(t,\psi)\|_{\mathbb{H}}^2\le L_b\|\gamma-\psi\|_{\mathfrak{B}_h}^2,$$

for $\gamma,\psi\in\mathfrak{B}_h$ and $t\in J$.

(H_2) The function $g\colon J \times \mathfrak{B}_h \to \mathcal{L}(\mathbb{K}, \mathbb{H})$ is continuous and there exists a non-negative constant L_g such that

$$E\|g(t,\gamma) - g(t,\phi)\|^2_{\mathcal{L}(\mathbb{K},\mathbb{H})} \le L_g \|\gamma - \phi\|^2_{\mathfrak{B}_h}.$$

(H_3) The functions $I_k\colon \mathbb{H} \to \mathbb{H}$ are continuous and there exists a positive constant L_I such that

$$E\|I_k(x) - I_k(y)\|^2_{\mathbb{H}} \le L_I E\|x - y\|^2_{\mathbb{H}}$$

for any $x, y \in \mathbb{H}$, $t \in J$ and each $k = 1, 2, \ldots, m$.

3 Existence and uniqueness result

Our first result on the existence and uniqueness of solutions to the system (1.1)–(1.3) is proved by using the Banach contraction principle.

Theorem 1 *Let the assumptions* (H_0)–(H_3) *hold and let us assume that*

$$\Delta = \left\{ 4m\widetilde{M}_S^2 L_I + 4l\left[\frac{1}{\alpha^2}\big(q^* L_b \widetilde{M}_T^2 T^{2\alpha}\big) + \widetilde{M}_S^2\big(L_f + L_g\big) \right] \right\} < 1,$$

where $q^* = \sup_{t\in[0,T]} \int_0^t \|q(t-s)\|\,\mathrm{d}s$. *Then the mild solution of the system* (1.1)–(1.3) *exists uniquely on* J.

Proof. We consider the operator $P\colon \mathfrak{B}'_h \to \mathfrak{B}'_h$ defined as

$$(Px)(t) = \begin{cases} \phi(t) \ \text{ for } \ t \le 0 \\ S_\alpha(t)\phi(0) + \displaystyle\int_0^t T_\alpha(t-s)\left[\int_0^s q(s-\eta)b(\eta, x_\eta)\,\mathrm{d}\eta\right]\mathrm{d}s \\ \quad + \displaystyle\int_0^t S_\alpha(t-s)f(s,x_s)\,\mathrm{d}s + \int_0^t S_\alpha(t-s)g(s,x_s)\,\mathrm{d}w(s), \quad t \in (0, t_1], \\ S_\alpha(t)\phi(0) + S_\alpha(t-t_1)I_1(x(t_1^-)) \\ \quad + \displaystyle\int_0^t T_\alpha(t-s)\left[\int_0^s q(s-\eta)b(\eta, x_\eta)\,\mathrm{d}\eta\right]\mathrm{d}s \\ \quad + \displaystyle\int_0^t S_\alpha(t-s)f(s,x_s)\,\mathrm{d}s + \int_0^t S_\alpha(t-s)g(s,x_s)\,\mathrm{d}w(s), \quad t \in (t_1, t_2], \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ S_\alpha(t)\phi(0) + \displaystyle\sum_{i=1}^m S_\alpha(t-t_i)I_i(x(t_i^-)) \\ \quad + \displaystyle\int_0^t T_\alpha(t-s)\left[\int_0^s q(s-\eta)b(\eta, x_\eta)\,\mathrm{d}\eta\right]\mathrm{d}s \\ \quad + \displaystyle\int_0^t S_\alpha(t-s)f(s,x_s)\,\mathrm{d}s + \int_0^t S_\alpha(t-s)g(s,x_s)\,\mathrm{d}w(s), \quad t \in (t_m, T]. \end{cases}$$

Let the function $y(\cdot)\colon (-\infty, T] \to \mathbb{H}$ be defined as

$$y(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ 0, & t \in J. \end{cases}$$

Then, $y_0 = \phi$. For every $z\colon J \to \mathbb{H}$ with $z|_{J_k} \in C(J_k, \mathbb{H})$, $k = 1, 2, \ldots, m$, and $z(0) = 0$ we define $\overline{z}$ by

$$\overline{z} = \begin{cases} 0, & t \in (-\infty, 0], \\ z(t), & t \in J. \end{cases}$$

If $x(\cdot)$ satisfies the system (2.1), then we can write $x(\cdot)$ as $x(t) = y(t) + \overline{z}(t)$, which implies that $x_t = y_t + \overline{z}_t$ for $t \in J$ and that $z(\cdot)$ satisfies

$$z(t) = \begin{cases} S_\alpha(t)\phi(0) + \int_0^t T_\alpha(t-s)\Big[\int_0^s q(s-\eta)b(\eta, y_\eta + \overline{z}_\eta)\,\mathrm{d}\eta\Big]\mathrm{d}s \\ \quad + \int_0^t S_\alpha(t-s)f(s, y_s + \overline{z}_s)\,\mathrm{d}s + \int_0^t S_\alpha(t-s)g(s, y_s + \overline{z}_s)\,\mathrm{d}w(s), & t \in (0, t_1], \\ S_\alpha(t)\phi(0) + S_\alpha(t-t_1)I_1(y(t_1^-) + \overline{z}(t_1^-)) \\ \quad + \int_0^t T_\alpha(t-s)\Big[\int_0^s q(s-\eta)b(\eta, y_\eta + \overline{z}_\eta)\,\mathrm{d}\eta\Big]\mathrm{d}s \\ \quad + \int_0^t S_\alpha(t-s)f(s, y_s + \overline{z}_s)\,\mathrm{d}s + \int_0^t S_\alpha(t-s)g(s, y_s + \overline{z}_s)\,\mathrm{d}w(s), & t \in (t_1, t_2], \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ S_\alpha(t)\phi(0) + \sum_{i=1}^m S_\alpha(t-t_i)I_i(y(t_i^-) + \overline{z}(t_i^-)) \\ \quad + \int_0^t T_\alpha(t-s)\Big[\int_0^s q(s-\eta)b(\eta, y_\eta + \overline{z}_\eta)\,\mathrm{d}\eta\Big]\mathrm{d}s \\ \quad + \int_0^t S_\alpha(t-s)f(s, y_s + \overline{z}_s)\,\mathrm{d}s + \int_0^t T_\alpha(t-s)g(s, y_s + \overline{z}_s)\,\mathrm{d}w(s), & t \in (t_m, T]. \end{cases}$$

Set $\mathfrak{B}_h'' = \{z \in \mathfrak{B}_h' : z_0 = 0\}$. For any $z \in \mathfrak{B}_h''$, we have

$$\|z\|_{\mathfrak{B}_h''} = \|z_0\|_{\mathfrak{B}_h} + \sup_{t\in J}(E\|z(t)\|^2)^{\frac{1}{2}} = \sup_{t\in J}(E\|z(t)\|^2)^{\frac{1}{2}}.$$

Hence, $(\mathfrak{B}''_h, \|\cdot\|_{\mathfrak{B}''_h})$ is a Banach space. Consider the operator $N\colon \mathfrak{B}''_h \to \mathfrak{B}''_h$ defined by

$$(Nz)(t)=\begin{cases} S_\alpha(t)\phi(0)+\displaystyle\int_0^t T_\alpha(t-s)\Big[\int_0^s q(s-\eta)b(\eta,y_\eta+\overline{z}_\eta)\,\mathrm{d}\eta\Big]\mathrm{d}s \\ \quad +\displaystyle\int_0^t S_\alpha(t-s)f(s,y_s+\overline{z}_s)\,\mathrm{d}s+\int_0^t S_\alpha(t-s)g(s,y_s+\overline{z}_s)\,\mathrm{d}w(s), \quad t\in(0,t_1], \\ S_\alpha(t)\phi(0)+S_\alpha(t-t_1)I_1(y(t_1^-)+\overline{z}(t_1^-)) \\ \quad +\displaystyle\int_0^t T_\alpha(t-s)\Big[\int_0^s q(s-\eta)b(\eta,y_\eta+\overline{z}_\eta)\,\mathrm{d}\eta\Big]\mathrm{d}s \\ \quad +\displaystyle\int_0^t S_\alpha(t-s)f(s,y_s+\overline{z}_s)\,\mathrm{d}s+\int_0^t S_\alpha(t-s)g(s,y_s+\overline{z}_s)\,\mathrm{d}w(s), \quad t\in(t_1,t_2], \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ S_\alpha(t)\phi(0)+\displaystyle\sum_{i=1}^m S_\alpha(t-t_i)I_i(y(t_i^-)+\overline{z}(t_i^-)) \\ \quad +\displaystyle\int_0^t T_\alpha(t-s)\Big[\int_0^s q(s-\eta)b(\eta,y_\eta+\overline{z}_\eta)\,\mathrm{d}\eta\Big]\mathrm{d}s \\ \quad +\displaystyle\int_0^t S_\alpha(t-s)f(s,y_s+\overline{z}_s)\,\mathrm{d}s+\int_0^t T_\alpha(t-s)g(s,y_s+\overline{z}_s)\,\mathrm{d}w(s), \quad t\in(t_m,T]. \end{cases}$$

To prove the existence result it is enough to show that N has a unique fixed point. For this, let $z, z^* \in \mathfrak{B}''_h$. Then, for any $t \in [0, t_1]$ we have

$$\begin{aligned} &E\|(Nz)(t)-(Nz^*)(t)\|^2_{\mathbb{H}} \\ &\quad\le 3E\Big\|\int_0^t T_\alpha(t-s)\Big[\int_0^s q(s-\eta)\{b(\eta,y_\eta+\overline{z}_\eta)-b(\eta,y_\eta+\overline{z}^*_\eta)\}\,\mathrm{d}\eta\Big]\mathrm{d}s\Big\|^2_{\mathbb{H}} \\ &\qquad+3E\Big\|\int_0^t S_\alpha(t-s)[f(s,y_s+\overline{z}_s)-f(s,y_s+\overline{z}^*_s)]\,\mathrm{d}s\Big\|^2_{\mathbb{H}} \\ &\qquad+3E\Big\|\int_0^t S_\alpha(t-s)[g(s,y_s+\overline{z}_s)-g(s,y_s+\overline{z}^*_s)]\,\mathrm{d}w(s)\Big\|^2_{\mathbb{H}} \\ &\quad\le\bigg(3l\Big[\frac{1}{\alpha^2}\big(q^*L_b\widetilde{M}_T^2T^{2\alpha}\big)+\widetilde{M}_S^2\big(L_f+L_g\big)\Big]\bigg)\|z-z^*\|^2_{\mathfrak{B}''_h}. \end{aligned}$$

Moreover, in view of the assumptions, for any $t \in (t_k, t_{k+1}]$, $k = 1, \ldots, m$, we obtain

$$\begin{aligned} &E\|(Nz)(t)-(Nz^*)(t)\|^2_{\mathbb{H}} \\ &\quad\le 4E\big\|S_\alpha(t-t_k)[I_k(y(t_k^-)+\overline{z}(t_k^-))-I_k(y(t_k^-)+\overline{z}^*(t_k^-))\big\|^2_{\mathbb{H}} \\ &\qquad+4E\Big\|\int_0^t T_\alpha(t-s)[f(s,y_s+\overline{z}_s)-f(s,y_s+\overline{z}^*_s)]\,\mathrm{d}s\Big\|^2_{\mathbb{H}} \\ &\qquad+4E\Big\|\int_0^t T_\alpha(t-s)\Big[\int_0^t q(s-\eta)\{b(\eta,y_\eta+\overline{z}_\eta)-b(\eta,y_\eta+\overline{z}^*_\eta)\}\,\mathrm{d}\eta\Big]\mathrm{d}s\Big\|^2_{\mathbb{H}} \\ &\qquad+4E\Big\|\int_0^t T_\alpha(t-s)[g(s,y_s+\overline{z}_s)-g(s,y_s+\overline{z}^*_s)]\,\mathrm{d}w(s)\Big\|^2_{\mathbb{H}} \\ &\quad\le\bigg(4m\widetilde{M}_S^2L_I+4l\Big[\frac{1}{\alpha^2}\big(q^*L_b\widetilde{M}_T^2T^{2\alpha}\big)+\widetilde{M}_S^2\big(L_f+L_g\big)\Big]\bigg)\|z-z^*\|^2_{\mathfrak{B}''_h}. \end{aligned}$$

Hence, for all $t \in [0, T]$ we have

$$\|(Nz)(t) - (Nz^*)(t)\|^2_{\mathfrak{B}''_h} \leq \Delta \|z - z^*\|^2_{\mathfrak{B}''_h}.$$

Since $\Delta < 1$, N is a contraction map. This implies that the operator N has a unique fixed point $z \in \mathfrak{B}''_h$, which is the mild solution of the system (1.1)–(1.3) on J. This completes the proof of the theorem. □

In our second result, we discuss the existence of mild solutions to (1.1)–(1.3) by using Krasnoselskii's fixed point theorem [17]. To prove the required result we consider the following assumptions.

(H_4) The functions $f, b\colon J \times \mathfrak{B}_h \to \mathbb{H}$ and $g\colon J \times \mathfrak{B}_h \to \mathcal{L}(\mathbb{K}, \mathbb{H})$ are continuous and there exist continuous functions $\mu_1, \mu_2, \mu_3\colon J \to (0, \infty)$ such that

$$E\|b(t, \phi)\|^2_{\mathbb{H}} \leq \mu_1(t)\|\phi\|^2_{\mathfrak{B}_h},$$
$$E\|f(t, \phi)\|^2_{\mathbb{H}} \leq \mu_2(t)\|\phi\|^2_{\mathfrak{B}_h},$$
$$E\|g(t, \phi)\|^2_{\mathcal{L}(\mathbb{K},\mathbb{H})} \leq \mu_3(t)\|\phi\|^2_{\mathfrak{B}_h},$$

where $\phi \in \mathfrak{B}_h$ and $t \in J$.

(H_5) The functions $I_k\colon \mathbb{H} \to \mathbb{H}$ are continuous and there exists a constant $\nabla > 0$ such that

$$\nabla = \max_{1\leq k\leq m,\, x\in B_p} \{E\|I_k(x)\|^2_{\mathbb{H}}\},$$

where B_p is the closed ball of radius p centered at zero, that is, $B_p = \{z \in \mathfrak{B}''_h : \|z\|^2_{\mathfrak{B}''_h} \leq p\}$, with radius p defined in the Proof of Theorem 2 below.

Theorem 2 *Let the assumptions* (H_0)–(H_2), (H_4) *and* (H_5) *be satisfied with*

$$\Omega = \left(3l\left[\frac{1}{\alpha^2}\big(q^* L_b \widetilde{M}_T^2 T^{2\alpha}\big) + \widetilde{M}_S^2\big(L_f + L_g\big)\right]\right) < 1. \tag{3.1}$$

Then the system (1.1)–(1.3) *has at least one mild solution on* J.

Proof. Choose

$$p \geq 4\left[\frac{1}{\alpha^2}\big(\widetilde{M}_T^2 T^{2\alpha}\omega_1\big) + \widetilde{M}_S^2\big[m\nabla + \omega_2 + \omega_3\big]\right],$$

where $\omega_1 = 4\mu_1^* q^*(\|\phi\|^2_{B_h} + l^2 p)$, $\omega_2 = 4\mu_2^*(\|\phi\|^2_{B_h} + l^2 p)$, $\omega_3 = 4\mu_3^*(\|\phi\|^2_{B_h} + l^2 p)$ and $\mu_1^* = \sup_{s\in[0,t]} \mu_1(s)$, $\mu_2^* = \sup_{s\in[0,t]} \mu_2(s)$, $\mu_3^* = \sup_{s\in[0,t]} \mu_3(s)$. Clearly, B_p is a bounded, closed and convex subset of $\mathfrak{B}''_h$. Let us consider the operators $\psi_1\colon B_p \to B_p$ and $\psi_2\colon B_p \to B_p$ defined as

$$(\psi_1 z)(t) = \begin{cases} 0, & t \in (0, t_1], \\ S_\alpha(t - t_1) I_1(y(t_1^-) + \overline{z}(t_1^-)), & t \in (t_1, t_2], \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ \displaystyle\sum_{i=1}^{m} S_\alpha(t - t_i) I_i(y(t_i^-) + \overline{z}(t_i^-)), & t \in (t_k, t_{k+1}], \end{cases}$$

and

$$(\psi_2 z)(t)=\begin{cases}\displaystyle\int_0^t T_\alpha(t-s)\Big[\int_0^s q(s-\eta)b(\eta,y_\eta+\overline{z}_\eta)\,\mathrm{d}\eta\Big]\mathrm{d}s\\ \displaystyle\quad+\int_0^t S_\alpha(t-s)f(s,y_s+\overline{z}_s)\,\mathrm{d}s+\int_0^t S_\alpha(t-s)g(s,y_s+\overline{z}_s)\,\mathrm{d}w(s),\quad t\in(0,t_1],\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ \displaystyle\int_0^t T_\alpha(t-s)\Big[\int_0^s q(s-\eta)b(\eta,y_\eta+\overline{z}_\eta)\,\mathrm{d}\eta\Big]\mathrm{d}s\\ \displaystyle\quad+\int_0^t S_\alpha(t-s)f(s,y_s+\overline{z}_s)\,\mathrm{d}s+\int_0^t S_\alpha(t-s)g(s,y_s+\overline{z}_s)\,\mathrm{d}w(s),\quad t\in(t_k,t_{k+1}].\end{cases}$$

We show that the operator ψ_1 is compact and continuous, and ψ_2 is a contraction. We divide the proof into the following five steps.

Step 1. First, we show that $\psi_1 z+\psi_2 z^*\in B_p$ for any $z,z^*\in B_p$. For $t\in(0,t_1]$, we have

$$E\|(\psi_1 z)(t)+(\psi_2 z^*)(t)\|^2_{\mathbb{H}}\le 3\left\{\frac{1}{\alpha^2}\big(\widetilde{M}_T^2T^{2\alpha}\omega_1\big)+\widetilde{M}_S^2[\omega_2+\omega_3]\right\}.$$

Similarly, for $t\in(t_i,t_{i+1}]$, $i=1,2,\dots,m$, we have

$$E\|(\psi_1 z)(t)+(\psi_2 z^*)(t)\|^2_{\mathbb{H}}\le 4\left\{\frac{1}{\alpha^2}\big(\widetilde{M}_T^2T^{2\alpha}\omega_1\big)+\widetilde{M}_S^2[m\nabla+\omega_2+\omega_3]\right\}\le p.$$

This implies that $\|(\psi_1 z)(t)+(\psi_2 z^*)(t)\|_{\mathfrak{B}_h''}\le p$, which means that $(\psi_1 z)(t)+(\psi_2 z^*)(t)\in B_p$.

Step 2. We will show that the mapping ψ_1 is continuous on B_p. For this we consider a convergent sequence $\{z^n\}_{n=1}^\infty$ in B_p with $\lim z^n\to z\in B_p$. Then, for $t\in(t_i,t_{i+1}]$, $i=1,\dots,m$, we have

$$E\|(\psi_1 z^n)(t)-(\psi_1 z)(t)\|^2_{\mathbb{H}}\le m\|S_\alpha(t-t_i)\|^2\big[E\|I_i(y(t_i^-)+\overline{z}^n(t_i^-))-I_i(y(t_i^-)+\overline{z}(t_i^-))\|^2_{\mathbb{H}}\big].$$

Since the functions I_i, $i=1,2,\dots,m$, are continuous, this implies that

$$\lim_{n\to\infty}E\|\psi_1 z^n-\psi_1 z\|^2_{\mathbb{H}}=0.$$

Therefore, the mapping ψ_1 is continuous on B_p.

Step 3. We will show that ψ_1 maps bounded sets into bounded sets in B_p. For this purpose we prove that there exists $\widehat{u}>0$ such that for each $z\in B_p$ we have $E\|(\psi_1 z)(t)\|^2_{\mathbb{H}}\le\widehat{u}$ for $t\in(t_i,t_{i+1}]$, $i=0,1,\dots,m$. Now, we have

$$\begin{aligned}E\|(\psi_1 z)(t)\|^2_{\mathbb{H}}&\le E\sum_{i=1}^m\|S_\alpha(t-t_i)I_i(y(t_i^-)+z(t_i^-))\|^2_{\mathbb{H}}\\&\le m\widetilde{M}_S^2\nabla=\widehat{u}.\end{aligned}$$

Step 4. In this step we show that the map ψ_1 is equicontinuous. Let $\tau_1,\tau_2\in(t_i,t_{i+1}]$, $t_i\le\tau_1<\tau_2\le t_{i+1}$, $i=0,1,2,\dots,m$. Then, for $z\in B_p$ we have

$$E\|(\psi_1 z)(\tau_2)-(\psi_1 z)(\tau_1)\|^2_{\mathbb{H}}\le m\|S_\alpha(\tau_2-t_i)-S_\alpha(\tau_1-t_i)\|^2E\|I_i(y(t_i^-)+z(t_i^-))\|^2_{\mathbb{H}}.$$

As $\tau_2 \to \tau_1$, and as S_α is strongly continuous, we obtain

$$\lim_{\tau_1 \to \tau_2} E\|(\psi_1 z)(\tau_2) - (\psi_1 z)(\tau_1)\|_{\mathbb{H}}^2 = 0,$$

which implies that ψ_1 is equicontinuous. On combining Step 2 to Step 4 together with Ascoli's theorem, we conclude that the operator ψ_1 is compact.

Step 5. Now, we prove that the mapping ψ_2 is a contraction. For this purpose, let $z, z^* \in B_p$ and let $t \in (t_i, t_{i+1}]$, where $i = 0, 1, \ldots, m$. Then, we have

$$\begin{aligned}
&E\|(\psi_2 z)(t) - (\psi_2 z^*)(t)\|_{\mathbb{H}}^2 \\
&\quad\le 3E\left\|\int_0^t T_\alpha(t-s)\left[\int_0^t q(s-\eta)\{b(\eta, y_\eta + \overline{z}_\eta) - b(\eta, y_\eta + \overline{z}_\eta^*)\}\,\mathrm{d}\eta\right]\mathrm{d}s\right\|_{\mathbb{H}}^2 \\
&\qquad + 3E\left\|\int_0^t S_\alpha(t-s)[f(s, y_s + \overline{z}_s) - f(s, y_s + \overline{z}_s^*)]\,\mathrm{d}s\right\|_{\mathbb{H}}^2 \\
&\qquad + 3E\left\|\int_0^t S_\alpha(t-s)[g(s, y_s + \overline{z}_s) - g(s, y_s + \overline{z}_s^*)]\,\mathrm{d}w(s)\right\|_{\mathbb{H}}^2 \\
&\quad\le \left(3l\left[\frac{1}{\alpha^2}\big(q^* L_b \widetilde{M}_T^2 T^{2\alpha}\big) + \widetilde{M}_S^2\big(L_f + L_g\big)\right]\right)\|z - z^*\|_{\mathfrak{B}_h''}^2
\end{aligned}$$

and

$$\|(\psi_2 z)(t) - (\psi_2 z^*)(t)\|_{\mathfrak{B}_h''}^2 \le \Omega\|z - z^*\|_{\mathfrak{B}_h''}^2.$$

As $\Omega < 1$ by the formula (3.1), we conclude that the mapping ψ_2 is of a contraction type. Therefore, by Krasnoselskii's fixed point theorem we deduce that the problem (1.1)–(1.3) admits at least one solution on J. Hence, the proof of the theorem is complete. □

4 Example

We consider the following impulsive partial differential equation of fractional order of the form:

$$\begin{aligned}
\frac{\partial^q u(t,x)}{\partial t^q} &= \frac{\partial^2 u(t,x)}{\partial x^2} + \int_0^t \cos(t-v)\left(\frac{1}{25}\int_{-\infty}^v B(v,x,s-v)Q_1(u(s,x))\,\mathrm{d}v\right)\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\left[\frac{1}{36}\int_{-\infty}^s F(s,x,\tau-s)Q_2(u(\tau,x))\,\mathrm{d}\tau\right. \qquad (4.1)\\
&\quad \left. + \left(\frac{1}{49}\int_{-\infty}^s G(s,x,\tau-s)Q_3(u(\tau,x))\,\mathrm{d}\tau\right)\frac{\mathrm{d}w(s)}{\mathrm{d}s}\right]\mathrm{d}s, \quad t \ne t_k,
\end{aligned}$$

$$\Delta u(t_i)(x) = \int_{-\infty}^{t_i} d_i(t_i - s)u(s,x)\,\mathrm{d}s, \quad x \in [0,\pi], \qquad (4.2)$$

$$u(t,0) = u(t,\pi) = 0, \quad t \ge 0, \qquad (4.3)$$

$$u(t,x) = \phi(t,x), \quad t \in (-\infty, 0], \qquad (4.4)$$

where $\frac{\partial^q}{\partial t^q}$ is Caputo's derivative of fractional order between 0 and 1, $0 < t_1 < t_2 < \cdots < t_m \le T$ are prefixed numbers, $\phi \in \mathfrak{B}_h$. Let $\mathbb{H} = L^2[0,\pi]$ and define the operator $A\colon D(A) \subset \mathbb{H} \to \mathbb{H}$ by

$A\omega = \omega''$. According to Theorem 3.1 in [6] the operator A is the infinitesimal generator of a solution operator $\{S_\alpha(t)\}_{t\geq 0}$. Since $S_\alpha(t)$ is strongly continuous on $[0,\infty)$, by the uniform boundedness principle, there exists a constant $M > 0$ such that $\|S_\alpha(t)\|_{L(\mathbb{H})} \leq M$ for $t \in [0,T]$. Here equation (4.2) denotes the impulsive condition, while (4.3) denotes the boundary conditions. Let $h(s) = e^{2s}$ for $s < 0$. Then $l = \int_{-\infty}^0 h(s)\,\mathrm{d}s = \frac{1}{2}$, and define

$$\|\phi\|_{\mathfrak{B}_h} = \int_{-\infty}^0 h(s) \sup_{\theta\in[s,0]} \|\phi(\theta)\|_{L^2}\,\mathrm{d}s.$$

Given a function $h\colon [0,\pi]\times[0,T] \to \mathbb{R}$ such that for each $t \in [0,T]$, $h(\cdot,t)\colon [0,\pi] \to \mathbb{H}$, we may identify it with the function $h\colon [0,\pi] \to \mathbb{H}$ given by $h(t)(x) = h(x,t)$. Set $u(t)(x) = u(t,x)$ and

$$b(t,\phi)(x) = \frac{1}{25}\int_{-\infty}^0 B(t,x,\theta)Q_1(\phi(\theta,t)(x))\,\mathrm{d}\theta,$$

$$f(t,\phi)(x) = \frac{1}{36}\int_{-\infty}^0 F(t,x,\theta)Q_2(\phi(\theta,t)(x))\,\mathrm{d}\theta,$$

$$g(t,\phi)(x) = \frac{1}{49}\int_{-\infty}^0 G(t,x,\theta)Q_3(\phi(\theta,t)(x))\,\mathrm{d}\theta,$$

$$I_i(\phi)(x) = \int_{-\infty}^0 d_i(-\theta)\phi(\theta)(x)\mathrm{d}\theta.$$

Then, with these settings equations (4.1)–(4.4) can be written in the abstract form (1.1)–(1.3). Furthermore, we have

$$\begin{aligned}
&\|b(t,\phi)(x) - b(t,\varphi)(x)\|_{L^2} \\
&\quad\leq \left[\int_0^\pi \left\{\frac{1}{25}\int_{-\infty}^0 B(t,x,\theta)\|Q_1(\phi(\theta,t)(x)) - Q_1(\varphi(\theta,t)(x))\|\,\mathrm{d}\theta\right\}^2 \mathrm{d}x\right]^{\frac{1}{2}} \\
&\quad\leq \frac{1}{25}\left[\int_0^\pi \left\{\int_{-\infty}^0 B(t,x,\theta)\,\mathrm{d}\theta\right\}^2 \mathrm{d}x\right]^{\frac{1}{2}} \|\phi - \varphi\|_{\mathfrak{B}_h},
\end{aligned}$$

and so

$$\|b(t,\phi)(x) - b(t,\varphi)(x)\|_{L^2} \leq \frac{C_1\sqrt{\pi}}{25}\|\phi - \varphi\|_{\mathfrak{B}_h}.$$

Similarly, we can find that $L_f = \frac{C_2\sqrt{\pi}}{36}$, $L_g = \frac{C_3\sqrt{\pi}}{49}$. Now, if we choose $m = 1$, $l = \alpha = \frac{1}{2}$, $q^* = 1$, $T = 1$ with $\widetilde{M}_S = \frac{1}{4}$, $\widetilde{M}_T = C_1 = C_2 = C_3 = 1$, we have

$$\Delta = \left\{4m\widetilde{M}_S^2 L_I + 4l\left[\frac{1}{\alpha^2}\big(q^* L_b \widetilde{M}_T^2 T^{2\alpha}\big) + \widetilde{M}_S^2\big(L_f + L_g\big)\right]\right\} = 0.41 < 1.$$

Therefore, the problem (4.1)–(4.4) has a unique mild solution u on $[0,1]$.

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