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EXISTENCE AND UNIQUENESS RESULTS FOR NONLINEAR ANISOTROPIC ELLIPTIC EQUATIONS

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Abstract. In this work, we shall be concerned with the existence and uniqueness of weak solutions of anisotropic elliptic operators $Au + \sum_{i=1}^{N} g_i(x,u,\nabla u) + \sum_{i=1}^{N} H_i(x,\nabla u) = f - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} k_i$, where the right-hand side f belongs to $L^{p'_\infty}(\Omega)$ and k_i belongs to $L^{p'_i}(\Omega)$ for $i = 1, \ldots, N$, where $p'_i = \frac{p_i}{p_i - 1}$, $p'_\infty = \frac{p_\infty}{p_\infty - 1}$ and $p_\infty = \max\{\overline{p}^*, p^+\}$ with $p^+ = \max\{p_1, \ldots, p_N\}$, $\overline{p} = \frac{1}{\frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i}}$, $\overline{p}^* = \frac{N\overline{p}}{N - \overline{p}}$, and A is a Leray–Lions operator. The critical growth condition on g_i is with respect to ∇u and no growth condition with respect to u is assumed; the function H_i grows as $|\nabla u|^{p_i - 1}$.

Keywords: Anisotropic elliptic equations, anisotropic Sobolev space, nonlinear operators.

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1 Introduction

We are interested in existence and uniqueness results for the following anisotropic quasi-linear elliptic problem

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x,\nabla u) + \sum_{i=1}^{N} g_i(x,u,\nabla u) + \sum_{i=1}^{N} H_i(x,\nabla u) = f - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} k_i & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is an open bounded domain with Lipschitz continuous boundary. The functions a_i, g_i and H_i are Carathéodory functions satisfying some conditions that we will specify in the following pages. The functions on the right-hand side f and k_i for $i = 1, \dots, N$ belong to suitable Lebesgue spaces. There is by now a large number of papers and an increasing interest in anisotropic problems with no hope of being complete. Let us mention some pioneering works on anisotropic Sobolev spaces: [3, 4, 8, 11, 12, 14, 15, 18, 20]. The existence of weak solutions or solutions to the problem (1.1) with $a_i(x, \nabla u) = |\frac{\partial u}{\partial x_i}|^{p_i-1}\frac{\partial u}{\partial x_i}$, $g_i \equiv 0$, $H_i \equiv 0$ and $k_i \equiv 0$ for $i = 1, \dots, N$ and right-hand side measure data was established by L. Boccardo *et al.* in [8]. An analogous existence result concerning the problem (1.1) for a system with $g_i \equiv 0$, $H_i \equiv 0$ and measure data was obtained by M. Bendahmane and Kenneth H. Karlsen in [5]. The problem (1.1) with $g_i \equiv 0$ and $H_i \neq 0$, when $f \in L^{p'_\infty}$ and $k_i \in L^{p'_i}(\Omega)$, where $p'_i = \frac{p_i}{p_i-1}$, $p'_\infty = \frac{p_\infty}{p_\infty-1}$ and $p_\infty = \max\{\overline{p}^*, p^+\}$ with $p^+ = \max\{p_1, \dots, p_N\}$, $\overline{p} = \frac{1}{\frac{1}{N}\sum_{i=1}^N \frac{1}{p_i}}$, $\overline{p}^* = \frac{N\overline{p}}{N-\overline{p}}$, was studied by R. Di Nardo and F. Feo in [14]. Moreover, the existence of weak solutions to the problem (1.1) with $H_i \equiv 0$ and $g_i \neq 0$ and $k_i \equiv 0$ was proved by Di Castro (see [13]).

Our objective is to study the problem (1.1) when $g_i \neq 0$ and $H_i \neq 0$ with $f \in L^{p'_\infty}$ and $k_i \in L^{p'_i}(\Omega)$.

Let us mention that many results in the isotropic case have been published for problems of the form (1.1) involving operators of type A in the variational case and in the L^1-data case. We restrict ourselves to papers dealing with k_i for $i = 0, \dots, N$ and f belonging to the dual. Since our problem is close to this case, we cite, among others, the papers of Guibé, Monetti and Randazzo [16, 19] and the recent work of Y. Akdim *et al.* [2]. The purpose of this paper is to establish the existence and uniqueness of weak solutions to some anisotropic elliptic equations with the two lower order terms. The proof of the existence of such solutions is based on techniques described, in particular, in [13, 14]. The uniqueness is obtained thanks to the following Lipschitz condition:

$$|g_i(x, s, \xi) - g_i(x, s', \xi')| \leq M|s - s'| + M\frac{|\xi_i - \xi'_i|}{(\eta + |\xi_i| + |\xi'_i|)^{\sigma_i}}$$

and

$$|H_i(x, \xi) - H_i(x, \xi')| \leq h\frac{|\xi_i - \xi'_i|}{(\eta + |\xi_i| + |\xi'_i|)^{\sigma_i}}$$

for some constants $h > 0$, $M > 0$, $\eta > 0$ and $\sigma_i > 0$ for $i = 1, \dots, N$.

The remaining part of this paper is organized as follows: Section 2 is devoted to preliminaries. In Section 3 we give some assumptions and definitions. The main existence results are stated and proved in Section 4. In Section 5 we prove the uniqueness results for the problem (1.1).

2 Preliminaries

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz continuous boundary and let $1 < p_1, \dots, p_N < \infty$ be N real numbers, $p^+ = \max\{p_1, \dots, p_N\}, p^- = \min\{p_1, \dots, p_N\}$ and $\overrightarrow{p} = (p_1, \dots, p_N)$.

The anisotropic Sobolev space (see [20])

$$W^{1,\overrightarrow{p}}(\Omega) = \left\{u \in W^{1,1}(\Omega) : \frac{\partial u}{\partial x_i} \in L^{p_i}(\Omega), i = 1, 2, \dots, N\right\}$$

is a Banach space with respect to the norm

$$\|u\|_{W^{1,\overrightarrow{p}}(\Omega)} = \|u\|_{L^1(\Omega)} + \sum_{i=1}^{N} \Big\| \frac{\partial u}{\partial x_i} \Big\|_{L^{p_i}(\Omega)}.$$

The space $W_0^{1,\overrightarrow{p}}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to this norm.

We recall a Poincaré-type inequality. Let $u \in W_0^{1,\overrightarrow{p}}(\Omega)$. Then for every $q \geq 1$ there exists a constant C_p (depending on q and p_i) such that (see [15])

$$\|u\|_{L^q(\Omega)} \leq C_p \Big\| \frac{\partial u}{\partial x_i} \Big\|_{L^q(\Omega)} \text{ for } i = 1, \dots, N. \tag{2.1}$$

Moreover, a Sobolev-type inequality holds. Let us denote by $\overline{p}$ the harmonic mean of these numbers, *i.e.*, $\frac{1}{\overline{p}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i}$. Let $u \in W_0^{1,\overrightarrow{p}}(\Omega)$. Then there exists (see [20]) a constant C_s such that

$$\|u\|_{L^q(\Omega)} \leq C_s \prod_{i=1}^{N} \Big\| \frac{\partial u}{\partial x_i} \Big\|_{L^{p_i}(\Omega)}^{\frac{1}{N}}, \tag{2.2}$$

where $q = \overline{p}^* = \frac{N\overline{p}}{N-\overline{p}}$ if $\overline{p} < N$, or $q \in [1, +\infty[$ if $\overline{p} \geq N$. It is possible to replace the geometric mean appearing on the right-hand side of (2.2) by the arithmetic mean. Indeed, let $a_1, \dots, a_N$ be positive numbers. Then

$$\prod_{i=1}^{N} a_i^{\frac{1}{N}} \leq \frac{1}{N} \sum_{i=1}^{N} a_i, \tag{2.3}$$

which, together with (2.2), implies that

$$\|u\|_{L^q(\Omega)} \leq \frac{C_s}{N} \sum_{i=1}^{N} \Big\| \frac{\partial u}{\partial x_i} \Big\|_{L^{p_i}(\Omega)}. \tag{2.4}$$

When

$$\overline{p} < N, \tag{2.5}$$

the inequality (2.4) implies the continuous embedding of the space $W_0^{1,\overrightarrow{p}}(\Omega)$ into $L^q(\Omega)$ for every $q \in [1, \overline{p}^*]$.

On the other hand, the continuity of the embedding $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^{p^+}(\Omega)$ relies on the inequality (2.1).

It may happen that $\overline{p}^* < p^+$, if the exponents p_i are not close enough. Then $p_\infty := \max\{\overline{p}*, p^+\}$ turns out to be the critical exponent in the anisotropic Sobolev embedding (see [15]).

Proposition 1 *If the condition* (2.5) *holds, then for* $q \in [1, p_\infty]$ *there is a continuous embedding* $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^q(\Omega)$. *For* $q < p_\infty$ *the embedding is compact.*

3 Assumptions and definitions

We consider the following class of nonlinear anisotropic elliptic homogeneous Dirichlet problems

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, \nabla u) + \sum_{i=1}^{N} g_i(x, u, \nabla u) + \sum_{i=1}^{N} H_i(x, \nabla u) = f - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} k_i & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ ($N \geq 2$) with Lipschitz continuous boundary $\partial\Omega$, $1 < p_1, \dots, p_N < \infty$ and (2.5) holds.

We assume that $a_i \colon \Omega \times \mathbb{R}^N \to \mathbb{R}$, $g_i \colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $H_i \colon \Omega \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory functions such that for all $s \in \mathbb{R}, \xi \in \mathbb{R}^N, \xi' \in \mathbb{R}^N$ and a.e. in Ω:

$$\sum_{i=1}^{N} a_i(x, \xi)\xi_i \geq \lambda \sum_{i=1}^{N} |\xi_i|^{p_i}, \tag{3.1}$$

$$|a_i(x, \xi)| \leq \gamma[j_i(x) + |\xi_i|^{p_i - 1}], \tag{3.2}$$

$$(a_i(x, \xi) - a_i(x, \xi'))(\xi_i - \xi_i') > 0 \quad \text{for } \xi_i \neq \xi_i', \tag{3.3}$$

$$g_i(x, s, \xi)s \geq 0, \tag{3.4}$$

$$|g_i(x, s, \xi)| \leq L(|s|)(c_i(x) + |\xi_i|^{p_i}) \quad \text{for all } i = 1, \dots, N, \tag{3.5}$$

$$|H_i(x, \xi)| \leq b_i |\xi_i|^{p_i - 1}, \tag{3.6}$$

where λ, γ, b_i are some positive constants, j_i is a positive function in $L^{p_i'}(\Omega)$, c_i is a positive function in $L^1(\Omega)$ for $i = 1, \dots, N$ and $L \colon \mathbb{R}^+ \to \mathbb{R}^+$is a continuous and non-decreasing function. Moreover, we suppose that

$$f \in L^{p_\infty'}(\Omega), \tag{3.7}$$

and

$$k_i \in L^{p_i'}(\Omega) \quad \text{for } i = 1, \dots, N. \tag{3.8}$$

Definition 1 *A function $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ is a weak solution to the problem* (1.1) *if $\sum_{i=1}^{N} g_i(x, u, \nabla u) \in L^1(\Omega)$ and u satisfies*

$$\sum_{i=1}^{N} \int_\Omega \left[a_i(x, \nabla u) \frac{\partial \varphi}{\partial x_i} + g_i(x, u, \nabla u)\varphi + H_i(x, \nabla u)\varphi \right] = \int_\Omega \left[f\varphi + \sum_{i=1}^{N} k_i \frac{\partial \varphi}{\partial x_i} \right]$$

for all $\varphi \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega)$.

4 Main results

In this section we prove the existence of at least a weak solution to the problem (1.1). The coercivity of the operator is guaranteed only if the norms of b_i are small enough. As usual we consider the approximate problems.

4.1 Approximate problems and a priori estimates

Let

$$g_i^n(x,u,\nabla u) = \frac{g_i(x,u,\nabla u)}{1+\frac{1}{n}|g_i(x,u,\nabla u)|} \quad \text{and} \quad H_i^n(x,\nabla u) = \frac{H_i(x,\nabla u)}{1+\frac{1}{n}|H_i(x,\nabla u)|}.$$

It is well-known (see e.g. [17]) that there exists at least a weak solution $u_n \in W_0^{1,\overrightarrow{p}}(\Omega)$ to the following problem

$$\begin{cases} -\displaystyle\sum_{i=1}^N \frac{\partial}{\partial x_i} a_i(x,\nabla u) + \sum_{i=1}^N g_i^n(x,u,\nabla u) + \sum_{i=1}^N H_i^n(x,\nabla u) = f - \sum_{i=1}^N \frac{\partial}{\partial x_i} k_i & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

The first and crucial step is an *a priori* estimate of u_n.

Lemma 1 *Let $A \in \mathbb{R}^+$ and $u \in W_0^{1,\overrightarrow{p}}(\Omega)$. Then there exists t measurable subsets $\Omega_1,\dots,\Omega_t$ of Ω and t functions $u_1,\dots,u_t$ such that $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$, $|\Omega_t| \leq A$ and $|\Omega_s| = A$ for $s \in \{1,\dots,t-1\}$, $\{x \in \Omega : |\frac{\partial u_s}{\partial x_i}| \neq 0$ for $i = 1,\dots,N\} \subset \Omega_s$, $\frac{\partial u}{\partial x_i} = \frac{\partial u_s}{\partial x_i}$ a.e. in Ω_s, $\frac{\partial(u_1+\ldots+u_s)}{\partial x_i} u_s = (\frac{\partial u}{\partial x_i}) u_s$, $u_1+\ldots+u_s = u$ in Ω and $\operatorname{sign}(u) = \operatorname{sign}(u_s)$ if $u_s \neq 0$ for $s \in \{1,\dots,t\}$ and $i \in \{1,\dots,N\}$.*

Proof. See [14, Lemma 4.2]. □

Proposition 2 *Assume that* (2.5), (3.1)–(3.8) *hold and let $u_n \in W_0^{1,\overrightarrow{p}}(\Omega)$ be a solution to the problem* (4.1)*. Then, we have*

$$\sum_{i=1}^N \int_\Omega \Big|\frac{\partial u_n}{\partial x_i}\Big|^{p_i} \leq C \tag{4.2}$$

for some positive constant C depending on N, Ω, λ, γ, p_i, b_i, $\|f\|_{L^{p'_\infty}(\Omega)}$, $\|g_i\|_{L^{p'_i}(\Omega)}$ for $i = 1,\dots,N$.

Proof. In what follows we do not explicitly write the dependence on n. Let A be a positive real number, that will be chosen later, referring to Lemma 1. Let us fix $s \in \{1,\dots,t\}$ and let us use $T_k(u_s)$ as a test function in the problem (4.1). Using (3.1), (3.4), Young's and Hölder's inequalities and Proposition 1 we obtain

$$\sum_{i=1}^N \int_{\{u_s \leq k\}} \Big|\frac{\partial u_s}{\partial x_i}\Big|^{p_i} \leq C_1 \Big(\|f\|_{L^{p'_\infty}(\Omega)} d_s^{\frac{1}{N}} + \sum_{i=1}^N \int_\Omega |H_i(x,\nabla u)||u_s| + \sum_{i=1}^N \|k_i\|^{p'_i}_{L^{p'_i}(\Omega)} \Big) \tag{4.3}$$

for some constant $C_1 > 0$, where

$$d_s = \prod_{i=1}^N \Big(\int_\Omega \Big|\frac{\partial u_s}{\partial x_i}\Big|^{p_i} \Big)^{\frac{1}{p_i}};$$

here and in what follows the constants depend on the data but not on the function u.

The dominated convergence theorem implies that

$$\sum_{i=1}^{N}\int_{\Omega}\Big|\frac{\partial u_s}{\partial x_i}\Big|^{p_i} \le C_1\bigg(\|f\|_{L^{p'_\infty}(\Omega)}d_s^{\frac{1}{N}} + \sum_{i=1}^{N}\int_{\Omega}|H_i(x,\nabla u)||u_s| + \sum_{i=1}^{N}\|k_i\|^{p'_i}_{L^{p'_i}(\Omega)}\bigg).$$

Using the condition (3.6), Hölder's and Young's inequalities, Lemma 1 and Proposition 1 we get

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega}|H_i(x,\nabla u)||u_s| \\
&\quad\le C_2\sum_{i=1}^{N}A^{\frac{1}{p_i}-\frac{1}{p_\infty}}\sum_{\sigma=1}^{s}\bigg[\frac{1}{p'_i}\int_{\Omega_\sigma}\Big|\frac{\partial u_\sigma}{\partial x_i}\Big|^{p_i} + \frac{CC_s}{p_i}\prod_{j=1}^{N}\Big\|\frac{\partial u_s}{\partial x_j}\Big\|^{\frac{p_i}{N}}_{L^{p_j}(\Omega)}\bigg] \quad \text{(see [14])} \qquad (4.4)\\
&\quad\le C_3\sum_{i=1}^{N}A^{\frac{1}{p_i}-\frac{1}{p_\infty}}\bigg[\int_{\Omega_s}\Big|\frac{\partial u_s}{\partial x_i}\Big|^{p_i} + \sum_{\sigma=1}^{s-1}\int_{\Omega_\sigma}\Big|\frac{\partial u_\sigma}{\partial x_i}\Big|^{p_i} + d_s^{\frac{p_i}{N}}\bigg]
\end{aligned}$$

for some constant $C_3 > 0$. Putting (4.4) in (4.3) we obtain

$$\begin{aligned}
\sum_{i=1}^{N}\int_{\Omega}\Big|\frac{\partial u_s}{\partial x_i}\Big|^{p_i} \le C_1\bigg\{&\|f\|_{L^{p'_\infty}(\Omega)}d_s^{\frac{1}{N}} + \sum_{i=1}^{N}\|k_i\|^{p'_i}_{L^{p'_i}(\Omega)} \\
&+ C_3\sum_{i=1}^{N}A^{\frac{1}{p_i}-\frac{1}{p_\infty}}\bigg[\int_{\Omega_s}\Big|\frac{\partial u_s}{\partial x_i}\Big|^{p_i} + \sum_{\sigma=1}^{s-1}\int_{\Omega_\sigma}\Big|\frac{\partial u_\sigma}{\partial x_i}\Big|^{p_i} + d_s^{\frac{p_i}{N}}\bigg]\bigg\}.
\end{aligned} \qquad (4.5)$$

If A is such that

$$1 - C_1C_3\sum_{i=1}^{N}A^{\frac{1}{p_i}-\frac{1}{p_\infty}} > 0, \qquad (4.6)$$

the inequality (4.5) becomes

$$\begin{aligned}
\sum_{i=1}^{N}\int_{\Omega}\Big|\frac{\partial u_s}{\partial x_i}\Big|^{p_i} \le C_4\bigg\{&\|f\|_{L^{p'_\infty}(\Omega)}d_s^{\frac{1}{N}} + \sum_{i=1}^{N}\|k_i\|^{p'_i}_{L^{p'_i}(\Omega)} \\
&+ \sum_{\sigma=1}^{s-1}\sum_{i=1}^{N}A^{\frac{1}{p_i}-\frac{1}{p_\infty}}\bigg(\sum_{j=1}^{N}\int_{\Omega_s}\Big|\frac{\partial u_s}{\partial x_j}\Big|^{p_j}\bigg) + \sum_{i=1}^{N}A^{\frac{1}{p_i}-\frac{1}{p_\infty}}d_s^{\frac{p_i}{N}}\bigg\}
\end{aligned} \qquad (4.7)$$

for some constant $C_4 > 0$, and for $s = 1$ we get

$$\int_{\Omega}\Big|\frac{\partial u_1}{\partial x_i}\Big|^{p_i} \le \sum_{i=1}^{N}\int_{\Omega}\Big|\frac{\partial u_1}{\partial x_i}\Big|^{p_i} \le C_4\bigg\{\|f\|_{L^{p'_\infty}(\Omega)}d_1^{\frac{1}{N}} + \sum_{i=1}^{N}\|k_i\|^{p'_i}_{L^{p'_i}(\Omega)} + \sum_{i=1}^{N}A^{\frac{1}{p_i}-\frac{1}{p_\infty}}d_1^{\frac{p_i}{N}}\bigg\}. \qquad (4.8)$$

Let us choose A such that (4.6) and $1 - C_4\sum_{i=1}^{N}A^{\frac{N}{p_i}\left(\frac{1}{p_i}-\frac{1}{p_\infty}\right)} > 0$ hold. We obtain (see [14])

$$d_1 = \prod_{i=1}^{N}\bigg(\int_{\Omega}\Big|\frac{\partial u_1}{\partial x_i}\Big|^{p_i}\bigg)^{\frac{1}{p_i}} \le C_5\bigg[\Big(\|f\|^{\frac{N}{\overline{p}}}_{L^{p'_\infty}(\Omega)} + \|\nu_2\|^{\frac{N}{\overline{p}}}_{L^{p'_\infty}(\Omega)}\Big)d_1^{\frac{1}{\overline{p}}} + \sum_{i=1}^{N}\|k_i\|^{p'_i}_{L^{p'_i}(\Omega)}\bigg].$$

Then there exists a constant $C_6 > 0$ such that $d_1 \leq C_6$ and by (4.8), we obtain

$$\sum_{i=1}^{N} \int_{\Omega} \Big| \frac{\partial u_1}{\partial x_i} \Big|^{p_i} \leq C_7 \tag{4.9}$$

for some constant $C_7 > 0$. Moreover, using (4.9) in (4.7) and iterating on s, we have

$$\sum_{i=1}^{N} \int_{\Omega} \Big| \frac{\partial u_s}{\partial x_i} \Big|^{p_i} \leq C_8 \left[\|f\|_{L^{p'_\infty}(\Omega)} d_s^{\frac{1}{N}} + \sum_{i=1}^{N} \|k_i\|_{L^{p'_i}(\Omega)}^{p'_i} + 1 + \sum_{i=1}^{N} A^{\frac{1}{p_i} - \frac{1}{p_\infty}} d_s^{\frac{p_i}{N}} \right].$$

Then, arguing as before, we obtain

$$\sum_{i=1}^{N} \int_{\Omega} \Big| \frac{\partial u_s}{\partial x_i} \Big|^{p_i} \leq C_9 \tag{4.10}$$

for some constant $C_9 > 0$. Thus,

$$\|u\|_{W_0^{1,\vec{p}}(\Omega)} \leq k \sum_{i=1}^{N} \left(\sum_{s=1}^{t} \int_{\Omega} \Big| \frac{\partial u_s}{\partial x_i} \Big|^{p_i} \right)^{\frac{1}{p_i}} \leq C_{10} \tag{4.11}$$

for some positive $k > 0$. □

Proposition 3 *If u_n is a weak solution of the problem* (4.1)*, then there exists a subsequence $(u_n)_n$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,\vec{p}}(\Omega)$, strongly in $L^{p_-}(\Omega)$ and a.e. in Ω.*

4.2 Strong convergence of $T_k(u_n)$

The following lemma generalizes to the anisotropic case the analogous Lemma 5 in [10]. We use the method of [1] and [10].

Lemma 2 *Assume that*

$$u_n \rightharpoonup u \text{ weakly in } W_0^{1,\vec{p}}(\Omega) \text{ and a.e. in } \Omega \tag{4.12}$$

and

$$\sum_{i=1}^{N} \int_{\Omega} \big[a_i(x, \nabla u_n) - a_i(x, \nabla u) \big] \left(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i} \right) \to 0. \tag{4.13}$$

Then

$$u_n \to u \text{ strongly in } W_0^{1,\vec{p}}(\Omega).$$

Proof. The proof follows as in Lemma 5 of [10] taking into account the anisotropy of the operator. □

Proposition 4 *Let u_n be a solution to the approximate problem* (4.1)*. Then*

$$T_k(u_n) \to T_k(u) \text{ strongly in } W_0^{1,\vec{p}}(\Omega).$$

Proof. Let us fix k and let δ be a real number such that $\delta \geq (\frac{L(k)}{2\lambda})^2$.

Let us define $z_n = T_k(u_n) - T_k(u)$ and $\varphi(s) = se^{\delta s^2}$. It is easy to check that for all $s \in \mathbb{R}$ one has

$$\varphi'(s) - \frac{L(k)}{\lambda}|\varphi(s)| \geq \frac{1}{2}. \tag{4.14}$$

Using $\varphi(z_n)$ as a test function in (4.1), we get

$$\begin{aligned} \sum_{i=1}^{N} \int_\Omega a_i(x, \nabla u_n) \frac{\partial \varphi(z_n)}{\partial x_i} + \sum_{i=1}^{N} \int_\Omega g_i^n(x, u_n, \nabla u_n)\varphi(z_n) + \sum_{i=1}^{N} \int_\Omega H_i^n(x, \nabla u_n)\varphi(z_n) \\ = \int_\Omega f\varphi(z_n) + \sum_{i=1}^{N} \int_\Omega k_i(x) \frac{\partial \varphi(z_n)}{\partial x_i}. \end{aligned} \tag{4.15}$$

Now, we investigate the convergence of every term in (4.15). Since $\varphi(z_n) \rightharpoonup 0$ weakly in $W_0^{1,\overrightarrow{p}}(\Omega)$, by Proposition 1, we have

$$\int_\Omega f\varphi(z_n) \to 0 \text{ as } n \to +\infty. \tag{4.16}$$

Since $|\varphi'(z_n)| \leq (1 + 8\delta k^2)e^{4\delta k^2}$, we infer that

$$\sum_{i=1}^{N} \int_\Omega k_i(x) \frac{\partial \varphi(z_n)}{\partial x_i} \to 0 \text{ as } n \to +\infty. \tag{4.17}$$

On the other hand,

$$\begin{aligned} \left| \sum_{i=1}^{N} \int_\Omega H_i^n(x, \nabla u_n)\varphi(z_n) \right| &\leq \sum_{i=1}^{N} \int_\Omega \left| \frac{\partial u_n}{\partial x_i} \right|^{p_i - 1} |b_i \varphi(z_n)| \\ &\leq \sum_{i=1}^{N} \left(\int_\Omega |b_i \varphi(z_n)|^{p_i} \right)^{\frac{1}{p_i}} \left(\int_\Omega \left| \frac{\partial u_n}{\partial x_i} \right|^{p_i} \right)^{\frac{1}{p_i'}} \\ &\leq C \sum_{i=1}^{N} \left(\int_\Omega |b_i \varphi(z_n)|^{p_i} \right)^{\frac{1}{p_i}}. \end{aligned}$$

By the dominated convergence theorem, we have $b_i\varphi(z_n) \to 0$ strongly in $L^{p_i}(\Omega)$. Then

$$\left| \sum_{i=1}^{N} \int_\Omega H_i^n(x, \nabla u_n)\varphi(z_n) \, \mathrm{d}x \right| \to 0 \text{ as } n \to +\infty. \tag{4.18}$$

In what follows we will denote by $\varepsilon_1(n), \varepsilon_2(n), \ldots$ various sequences of real numbers which converge to zero when n tends to $+\infty$.

Since $g_i^n(x, u_n, \nabla u_n)\varphi(z_n) \geq 0$ on the set $\{|u_n| > k\}$, by (4.15), (4.16), (4.17) and (4.18), we deduce that

$$\sum_{i=1}^{N} \int_\Omega a_i(x, \nabla u_n) \frac{\partial \varphi(z_n)}{\partial x_i} + \sum_{i=1}^{N} \int_{\{|u_n| \leq k\}} g_i^n(x, u_n, \nabla u_n)\varphi(z_n)dx \leq \varepsilon_1(n). \tag{4.19}$$

On the other hand, we get

$$\sum_{i=1}^{N}\int_{\Omega} a_i(x,\nabla u_n)\frac{\partial\varphi(z_n)}{\partial x_i} = \sum_{i=1}^{N}\int_{\Omega}\big(a_i(x,\nabla T_k(u_n)) - a_i(x,\nabla T_k(u))\big) \Big(\frac{\partial T_k(u_n)}{\partial x_i} - \frac{\partial T_k(u)}{\partial x_i}\Big)\varphi'(z_n)\,\mathrm{d}x + \varepsilon_2(n). \tag{4.20}$$

Indeed, we have

$$\begin{aligned}\sum_{i=1}^{N}\int_{\Omega} a_i(x,\nabla u_n)\frac{\partial\varphi(z_n)}{\partial x_i} &= \sum_{i=1}^{N}\int_{\Omega} a_i(x,\nabla u_n)\Big(\frac{\partial T_k(u_n)}{\partial x_i} - \frac{\partial T_k(u)}{\partial x_i}\Big)\varphi'(z_n)\\ &= \sum_{i=1}^{N}\int_{\Omega} a_i(x,\nabla T_k(u_n))\Big(\frac{\partial T_k(u_n)}{\partial x_i} - \frac{\partial T_k(u)}{\partial x_i}\Big)\varphi'(z_n)\\ &\quad - \sum_{i=1}^{N}\int_{\{u_n>k\}} a_i(x,\nabla u_n)\frac{\partial T_k(u)}{\partial x_i}\varphi'(z_n).\end{aligned}$$

The sequence $\big(a_i(x,\nabla u_n)\varphi'(z_n)\big)_n$ is bounded in $L^{p_i'}(\Omega)$. Then, since $\frac{\partial T_k(u)}{\partial x_i}\chi_{\{|u_n|>k\}} \to 0$ strongly in $L^{p_i}(\Omega)$, one has

$$\sum_{i=1}^{N}\int_{\Omega} a_i(x,\nabla u_n)\frac{\partial\varphi(z_n)}{\partial x_i}\,\mathrm{d}x = \sum_{i=1}^{N}\int_{\Omega} a_i(x,\nabla T_k(u_n))\Big(\frac{\partial T_k(u_n)}{\partial x_i} - \frac{\partial T_k(u)}{\partial x_i}\Big)\varphi'(z_n) + \varepsilon_3(n),$$

which we can rewrite as

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega} a_i(x,\nabla u_n)\frac{\partial\varphi(z_n)}{\partial x_i}\,\mathrm{d}x\\ &\quad= \sum_{i=1}^{N}\int_{\Omega}\big(a_i(x,\nabla T_k(u_n)) - a_i(x,\nabla T_k(u))\big)\Big(\frac{\partial T_k(u_n)}{\partial x_i} - \frac{\partial T_k(u)}{\partial x_i}\Big)\varphi'(z_n)\,\mathrm{d}x\\ &\qquad+ \sum_{i=1}^{N}\int_{\Omega} a_i(x,\nabla T_k(u))\Big(\frac{\partial T_k(u_n)}{\partial x_i} - \frac{\partial T_k(u)}{\partial x_i}\Big)\varphi'(z_n)\,\mathrm{d}x + \varepsilon_3(n).\end{aligned}$$

By Proposition 3, the growth condition (3.2) and Vitali's theorem one has $a_i(x,\nabla T_k(u))\varphi'(z_n) \to a_i(x,\nabla T_k(u))$ strongly in $L^{p_i'}(\Omega)$. Since $\frac{\partial T_k(u_n)}{\partial x_i} \rightharpoonup \frac{\partial T_k(u)}{\partial x_i}$ weakly in $L^{p_i}(\Omega)$, we have

$$\lim_{n\to+\infty}\sum_{i=1}^{N}\int_{\Omega} a_i(x,\nabla T_k(u))\Big(\frac{\partial T_k(u_n)}{\partial x_i} - \frac{\partial T_k(u)}{\partial x_i}\Big)\varphi'(z_n)\,\mathrm{d}x = 0.$$

Hence, we get (4.20).

On the other hand, we have

$$\begin{aligned}&\Big|\sum_{i=1}^{n}\int_{\{|u_n|\le k\}} g_i^n(x,u_n,\nabla u_n)\varphi(z_n)\,\mathrm{d}x\Big|\\ &\quad\le \frac{L(k)}{\lambda}\sum_{i=1}^{N}\int_{\Omega}\big(a_i(x,\nabla T_k(u_n))\\ &\qquad - a_i(x,\nabla T_k(u))\big)\Big(\frac{\partial T_k(u_n)}{\partial x_i} - \frac{\partial T_k(u)}{\partial x_i}\Big)|\varphi(z_n)|\,\mathrm{d}x + \varepsilon_4(n).\end{aligned} \tag{4.21}$$

Indeed, by virtue of (3.1), (3.5) and the fact that $\varphi(z_n) \to 0$ weakly* in $L^\infty(\Omega)$, we have

$$\begin{aligned}
&\left|\sum_{i=1}^N \int_{\{|u_n|\le k\}} g_i^n(x,u_n,\nabla u_n)\varphi(z_n)\,\mathrm{d}x\right| \\
&\quad\le L(k)\left(\sum_{i=1}^N \int_{\{|u_n|\le k\}} c_i|\varphi(z_n)| + \sum_{i=1}^N \int_{\{|u_n|\le k\}} \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i}|\varphi(z_n)|\,\mathrm{d}x\right) \\
&\quad\le L(k)\sum_{i=1}^N \int_\Omega \left|\frac{\partial T_k u_n}{\partial x_i}\right|^{p_i}|\varphi(z_n)|\,\mathrm{d}x + \varepsilon_4(n) \\
&\quad\le \frac{L(k)}{\lambda}\sum_{i=1}^N \int_\Omega a_i(x,\nabla T_k(u_n)).\frac{\partial T_k(u_n)}{\partial x_i}|\varphi(z_n)|\,\mathrm{d}x + \varepsilon_4(n) \\
&\quad\le \frac{L(k)}{\lambda}\sum_{i=1}^N \int_\Omega \big(a_i(x,\nabla T_k(u_n)) - a_i(x,\nabla T_k(u))\big) \\
&\qquad\left(\frac{\partial T_k(u_n)}{\partial x_i} - \frac{\partial T_k(u)}{\partial x_i}\right)|\varphi(z_n)|\,\mathrm{d}x \\
&\qquad+ \frac{L(k)}{\lambda}\sum_{i=1}^N \int_\Omega a_i(x,\nabla T_k(u_n)).\frac{\partial T_k(u)}{\partial x_i}|\varphi(z_n)|\,\mathrm{d}x \\
&\qquad+ \frac{L(k)}{\lambda}\sum_{i=1}^N \int_\Omega a_i(x,\nabla T_k(u)).\left(\frac{\partial T_k(u_n)}{\partial x_i} - \frac{\partial T_k(u)}{\partial x_i}\right)|\varphi(z_n)|\,\mathrm{d}x + \varepsilon_5(n).
\end{aligned}$$

Since $\frac{\partial T_k(u_n)}{\partial x_i} \rightharpoonup \frac{\partial T_k(u)}{\partial x_i}$ weakly in $L^{p_i}(\Omega)$ and $(\varphi(z_n))_n$ is bounded, we obtain

$$\sum_{i=1}^N \int_\Omega a_i(x,\nabla T_k(u)).\left(\frac{\partial T_k(u_n)}{\partial x_i} - \frac{\partial T_k(u)}{\partial x_i}\right)\varphi(z_n)\,\mathrm{d}x \to 0 \text{ as } n \to +\infty.$$

Thanks to (3.2) and (4.2), the sequence $\big(a_i(x,\nabla T_k(u_n))\big)_n$ is bounded in $L^{p_i'}(\Omega)$, so there exists $l_k^i \in L^{p_i'}(\Omega)$ such that $a_i(x,\nabla T_k(u_n)) \rightharpoonup l_k^i$ weakly in $L^{p_i'}(\Omega)$. Since $\varphi(z_n) \to 0$ weakly* in $L^\infty(\Omega)$, we conclude that

$$\sum_{i=1}^N \int_\Omega a_i(x,\nabla T_k(u_n))\frac{\partial T_k(u)}{\partial x_i}\varphi(z_n)\,\mathrm{d}x \to 0 \text{ as } n \to +\infty.$$

Hence, we get (4.21). Therefore, by combining (4.19), (4.20) and (4.21), we obtain

$$\begin{aligned}
&\sum_{i=1}^N \int_\Omega \big(a_i(x,\nabla T_k(u_n)) - a_i(x,\nabla T_k(u))\big) \\
&\qquad\left(\frac{\partial T_k(u_n)}{\partial x_i} - \frac{\partial T_k(u)}{\partial x_i}\right)\left(\varphi'(z_n) - \frac{L(k)}{\lambda}|\varphi(z_n)|\right)\,\mathrm{d}x \le \varepsilon_6(n).
\end{aligned} \tag{4.22}$$

By (4.14) and (4.22), we get

$$0 \le \sum_{i=1}^N \int_\Omega \big(a_i(x,\nabla T_k(u_n)) - a_i(x,\nabla T_k(u))\big)\left(\frac{\partial T_k(u_n)}{\partial x_i} - \frac{\partial T_k(u)}{\partial x_i}\right) \le 2\varepsilon_6(n).$$

Then Lemma 2 gives that $T_k(u_n) \to T_k(u)$ strongly in $W_0^{1,\vec{p}}(\Omega)$. □

4.3 Existence

Theorem 1 *Assume that* (2.5) *and* (3.1)–(3.8) *hold. Then there exists at least a weak solution to the problem* (1.1).

Proof. By (4.2) the sequence $(\frac{\partial u_n}{\partial x_i})_n$ is bounded in $L^{p_i}(\Omega)$, so we have that

$$\frac{\partial u_n}{\partial x_i} \rightharpoonup \frac{\partial u}{\partial x_i} \text{ weakly in } L^{p_i}(\Omega) \text{ for } i = 1, \dots, N, \tag{4.23}$$

$$u_n \to u \text{ strongly in } L^{p_-}(\Omega). \tag{4.24}$$

By Proposition 4 there exists a subsequence, which we still denote by u_n, such that

$$\frac{\partial u_n}{\partial x_i} \to \frac{\partial u}{\partial x_i} \quad \text{a.e. in } \Omega \quad \text{for } i = 1, \dots, N. \tag{4.25}$$

Then for $i = 1, \dots, N$ we have

$$\begin{cases} a_i(x, \nabla u_n) \to a_i(x, \nabla u) \text{ a.e. in } \Omega, \\ g_i^n(x, u_n, \nabla u_n) \to g_i(x, u, \nabla u) \text{ a.e. in } \Omega, \\ H_i^n(x, \nabla u_n) \to H_i(x, \nabla u) \text{ a.e. in } \Omega. \end{cases}$$

Moreover, by (3.2) and (3.6), we have

$$\int_\Omega |a_i(x, \nabla u_n)|^{p_i'} \leq C\left[\int_\Omega j_i^{p_i'} + \int_\Omega \Big|\frac{\partial u_n}{\partial x_i}\Big|^{p_i}\right] \quad \text{and} \quad \int_\Omega |H_i^n(x, \nabla u_n)|^{p_i'} \leq C \int_\Omega \Big|\frac{\partial u_n}{\partial x_i}\Big|^{p_i}.$$

By (4.2), $\big(a_i(x, \nabla u_n)\big)_n$ and $\big(H_i(x, \nabla u_n)\big)_n$ are bounded in $L^{p_i'}(\Omega)$. Then $a_i(x, \nabla u_n) \rightharpoonup a_i(x, \nabla u)$ weakly in $L^{p_i'}(\Omega)$ and $H_i(x, \nabla u_n) \rightharpoonup H_i(x, \nabla u)$ weakly in $L^{p_i'}(\Omega)$. Now, as in [13], we prove that $g_i^n(x, u_n, \nabla u_n)$ is uniformly equi-integrable for $i = 1, \dots, N$. If we take $T_k(u_n)$ as a test function in (4.1), by the Hölder inequality we get

$$\sum_{i=1}^N \int_\Omega g_i^n(x, u_n, \nabla u_n) T_k(u_n) \leq C_1.$$

Let E be a measurable subset of Ω. For any $k \in \mathbb{R}^+$, we have

$$\begin{aligned} &\int_E |g_i^n(x, u_n, \nabla u_n)| \,\mathrm{d}x \\ &\quad = \int_{E \cap \{|u_n| \leq k\}} |g_i^n(x, u_n, \nabla u_n)| \,\mathrm{d}x + \int_{E \cap \{|u_n| > k\}} |g_i^n(x, u_n, \nabla u_n)| \,\mathrm{d}x \\ &\quad \leq \int_{E \cap \{|u_n| \leq k\}} L(k) c_i(x) + \int_{E \cap \{|u_n| \leq k\}} L(k) \Big|\frac{\partial u_n}{\partial x_i}\Big|^{p_i} \mathrm{d}x + \int_{E \cap \{|u_n| > k\}} |g_i^n(x, u_n, \nabla u_n)| \,\mathrm{d}x \\ &\quad \leq \int_E L(k) c_i(x) + \int_E L(k) \Big|\frac{\partial T_k(u_n)}{\partial x_i}\Big|^{p_i} \mathrm{d}x + \frac{1}{k} \int_E T_k(u_n) g_i^n(x, u_n, \nabla u_n) \,\mathrm{d}x. \end{aligned}$$

Using the fact that $\frac{\partial T_k(u_n)}{\partial x_i} \to \frac{\partial T_k(u)}{\partial x_i}$ strongly in $L^{p_i}(\Omega)$ and

$$\int_E T_k(u_n) g_i^n(x, u_n, \nabla u_n) \,\mathrm{d}x \leq C_1,$$

we infer that g_i^n is uniformly equi-integrable for any i. Since $g_i^n(x, u_n, \nabla u_n) \to g_i(x, u, \nabla u)$ a.e. in Ω, thanks to Vitali's theorem, we get $g_i^n(x, u_n, \nabla u_n) \to g_i(x, u, \nabla u)$ in $L^1(\Omega)$.

That allows us to pass to the limit in the problem (4.1). □

Remark 1 *For the existence of weak solutions, the condition* (3.2) *can replaced by*

$$|a_i(x, s, \xi)| \leq \gamma \Big[j_i + |s|^{\frac{p_\infty}{p_i'}} + |\xi_i|^{p_i - 1} \Big],$$

where j_i is a positive function in $L^{p_i'}(\Omega)$ for $i = 1, \dots, N$.

5 Uniqueness

In this section we prove the uniqueness of weak solutions to the problem (1.1). Let us assume that a_i are strongly monotone:

$$(a_i(x, \xi) - a_i(x, \xi'))(\xi - \xi') \geq \alpha\big(\varepsilon + |\xi_i| + |\xi_i'|\big)^{p_i - 2} |\xi_i - \xi_i'|^2 \tag{5.1}$$

with $\alpha > 0$ and $\varepsilon \geq 0$.

The first uniqueness result is obtained when every p_i is not greater than 2, assuming the following Lipschitz conditions on g_i and H_i:

$$|g_i(x, s, \xi) - g_i(x, s', \xi')| \leq M|s - s'| + M \frac{|\xi_i - \xi_i'|}{\big(\eta + |\xi_i| + |\xi_i'|\big)^{\sigma_i}} \tag{5.2}$$

and

$$|H_i(x, \xi) - H_i(x, \xi')| \leq h \frac{|\xi_i - \xi_i'|}{\big(\eta + |\xi_i| + |\xi_i'|\big)^{\sigma_i}} \tag{5.3}$$

for some constants $h > 0$, $M > 0$, $\eta > 0$ and $\sigma_i > 0$ for $i = 1, \dots, N$.

Theorem 2 *Let $1 < p_i \leq 2$ if $N = 2$, $\frac{2N}{N+2} \leq p_i \leq 2$ if $N \geq 3$ and $\sigma_i \geq 1 - \frac{p_i}{2}$ for $i = 1, \dots, N$. Let us assume that* (2.5), (3.1)–(3.8), (5.1) *with $\varepsilon = 0$ and* (5.2), (5.3) *with $\eta > 0$ hold. Then there exists a unique weak solution to the problem* (1.1)

Proof. Following [3], let us suppose u and v are two weak solutions to the problem (1.1) and denote $w = (u - v)^+$ and $E_t = \{x \in \Omega : t < w < \sup w\}$ for $t \in [0, \sup w[$. We use

$$w_t = \begin{cases} w(x) - t, & \text{if } w(x) > t, \\ 0, & \text{otherwise} \end{cases} \tag{5.4}$$

as a test function in the difference of the equations. The strong monotonicity (5.1) with $\varepsilon = 0$ and the Lipschitz conditions (5.2) and (5.3) with $\eta > 0$ give

$$\begin{aligned} \sum_{i=1}^N \int_{E_t} \frac{\big|\frac{\partial w_t}{\partial x_i}\big|^2}{\big(\big|\frac{\partial u}{\partial x_i}\big| + \big|\frac{\partial v}{\partial x_i}\big|\big)^{2 - p_i}} \leq \frac{h}{\alpha} \sum_{i=1}^N \int_{E_t} \frac{\big|\frac{\partial w_t}{\partial x_i}\big| w_t}{\big(\eta + \big|\frac{\partial u}{\partial x_i}\big| + \big|\frac{\partial v}{\partial x_i}\big|\big)^{\sigma_i}} \\ + \frac{M}{\alpha} \sum_{i=1}^N \int_{E_t} \frac{\big|\frac{\partial w_t}{\partial x_i}\big| w_t}{\big(\eta + \big|\frac{\partial u}{\partial x_i}\big| + \big|\frac{\partial v}{\partial x_i}\big|\big)^{\sigma_i}} + \frac{MN}{\alpha} \int_{E_t} w_t w. \end{aligned}$$

Since $\sigma_i \geq 1 - \frac{p_i}{2}$, by the Young inequality and some easy computations we have

$$\sum_{i=1}^{N} \int_{E_t} \frac{\left|\frac{\partial w_t}{\partial x_i}\right|^2}{\left(\left|\frac{\partial u}{\partial x_i}\right| + \left|\frac{\partial v}{\partial x_i}\right|\right)^{2-p_i}} \leq C_1 \int_{E_t} w_t^2 + C_1 \int_{E_t} w^2 \tag{5.5}$$

for some positive constant C independent of t.

Since $\frac{\partial w}{\partial x_i} = \frac{\partial w_t}{\partial x_i}$, by (2.2) we get

$$\frac{1}{C_s}\left(\int_{E_t} w_t^2\right)^{\frac{1}{2}} \leq \prod_{i=1}^{N}\left(\int_{E_t} \left|\frac{\partial w_t}{\partial x_i}\right|^{\frac{2N}{N+2}}\right)^{\frac{N+2}{2N^2}} \quad \text{and} \quad \frac{1}{C_s}\left(\int_{E_t} w^2\right)^{\frac{1}{2}} \leq \prod_{i=1}^{N}\left(\int_{E_t} \left|\frac{\partial w_t}{\partial x_i}\right|^{\frac{2N}{N+2}}\right)^{\frac{N+2}{2N^2}}. \tag{5.6}$$

Then by (5.6), (2.3) and the Hölder inequality, we obtain

$$\frac{1}{C_s^2}\int_{E_t} w_t^2 \leq \frac{1}{N^2}\sum_{i=1}^{N} \int_{E_t} \frac{\left|\frac{\partial w_t}{\partial x_i}\right|^2}{\left(\left|\frac{\partial u}{\partial x_i}\right| + \left|\frac{\partial v}{\partial x_i}\right|\right)^{2-p_i}} \sum_{i=1}^{N}\left(\int_{E_t}\left(\left|\frac{\partial u}{\partial x_i}\right| + \left|\frac{\partial v}{\partial x_i}\right|\right)^{(2-p_i)\frac{N}{2}}\right)^{\frac{2}{N}}$$

and

$$\frac{1}{C_s^2}\int_{E_t} w^2 \leq \frac{1}{N^2}\sum_{i=1}^{N} \int_{E_t} \frac{\left|\frac{\partial w_t}{\partial x_i}\right|^2}{\left(\left|\frac{\partial u}{\partial x_i}\right| + \left|\frac{\partial v}{\partial x_i}\right|\right)^{2-p_i}} \sum_{i=1}^{N}\left(\int_{E_t}\left(\left|\frac{\partial u}{\partial x_i}\right| + \left|\frac{\partial v}{\partial x_i}\right|\right)^{(2-p_i)\frac{N}{2}}\right)^{\frac{2}{N}}.$$

Hence

$$\frac{1}{C_s^2}\sum_{i=1}^{N} \int_{E_t} \frac{\left|\frac{\partial w_t}{\partial x_i}\right|^2}{\left(\left|\frac{\partial u}{\partial x_i}\right| + \left|\frac{\partial v}{\partial x_i}\right|\right)^{2-p_i}} \leq \frac{2C_1}{N^2}\sum_{i=1}^{N} \int_{E_t} \frac{\left|\frac{\partial w_t}{\partial x_i}\right|^2}{\left(\left|\frac{\partial u}{\partial x_i}\right| + \left|\frac{\partial v}{\partial x_i}\right|\right)^{2-p_i}} \sum_{i=1}^{N}\left(\int_{E_t}\left(\left|\frac{\partial u}{\partial x_i}\right| + \left|\frac{\partial v}{\partial x_i}\right|\right)^{(2-p_i)\frac{N}{2}}\right)^{\frac{2}{N}}.$$

Therefore,

$$\frac{1}{C_s^2} \leq \frac{2C_1}{N^2}\sum_{i=1}^{N}\left(\int_{E_t}\left(\left|\frac{\partial u}{\partial x_i}\right| + \left|\frac{\partial v}{\partial x_i}\right|\right)^{(2-p_i)\frac{N}{2}}\right)^{\frac{2}{N}}.$$

Since $(2-p_i)\frac{N}{2} \leq p_i$, the dominated convergence theorem gives

$$\lim_{t \to \sup w} \int_{E_t}\left(\left|\frac{\partial u}{\partial x_i}\right| + \left|\frac{\partial v}{\partial x_i}\right|\right)^{(2-p_i)\frac{N}{2}} = 0,$$

which leads to a contradiction. □

The second result is obtained when every p_i is greater than 2 but $\varepsilon = 0$ in (5.1), and we assume the following Lipschitz conditions on g_i and H_i:

$$|g_i(x,s,\xi) - g_i(x,s',\xi')| \leq M|s-s'| + G_i(x)\big(|\xi_i| + |\xi_i'|\big)^{\sigma_i}|\xi_i - \xi_i'| \tag{5.7}$$

$$|H_i(x,\xi) - H_i(x,\xi')| \leq h_i(x)\big(|\xi_i| + |\xi_i'|\big)^{\sigma_i}|\xi_i - \xi_i'| \tag{5.8}$$

with $\sigma_i \geq 0$, $M \geq 0$, $G_i \in L^{s_i}(\Omega)$, $h_i \in L^{s_i}(\Omega)$ and $s_i \geq \frac{p_\infty p_i}{p_\infty - p_i}$.

Theorem 3 *Let us suppose that* $N \geq 3$, $2 \leq p_i \leq \frac{2Ns_i}{Ns_i - 2s_i + 2N}$, $s_i \geq \max\{N, \frac{p_\infty p_i}{p_\infty - p_i}\}$ *and* $0 \leq \sigma_i \leq \frac{p_i}{N} - \frac{p_i}{s_i} + \frac{p_i - 2}{2}$ *for* $i = 1, \ldots, N$. *Let us assume that* (2.5), (3.1)–(3.8), (5.1) *with* $\varepsilon > 0$, (5.7) *and* (5.8) *hold. Then there exists a unique weak solution to the problem* (1.1).

Proof. Arguing as in the proof of Theorem 2, by the strong monotonicity (5.1) with $\varepsilon = 0$ and the Lipschitz conditions (5.2) and (5.3), we get

$$\begin{aligned} &\sum_{i=1}^{N} \int_{E_t} \Big|\frac{\partial w_t}{\partial x_i}\Big|^2 \Big(\varepsilon + \Big|\frac{\partial u}{\partial x_i}\Big| + \Big|\frac{\partial v}{\partial x_i}\Big|\Big)^{p_i - 2} \\ &\quad \leq \frac{1}{\alpha} \sum_{i=1}^{N} \int_{E_t} h_i \Big(\Big|\frac{\partial u}{\partial x_i}\Big| + \Big|\frac{\partial v}{\partial x_i}\Big|\Big)^{\sigma_i} \Big|\frac{\partial w_t}{\partial x_i}\Big| w_t \\ &\qquad + \frac{1}{\alpha} \sum_{i=1}^{N} \int_{E_t} G_i \Big(\Big|\frac{\partial u}{\partial x_i}\Big| + \Big|\frac{\partial v}{\partial x_i}\Big|\Big)^{\sigma_i} \Big|\frac{\partial w_t}{\partial x_i}\Big| w_t + \frac{M}{\alpha} \sum_{i=1}^{N} \int_{E_t} w.w_t. \end{aligned} \tag{5.9}$$

If $\sigma_i \geq \frac{p_i - 2}{2}$, by (5.8), Young's and Hölder's inequalities, we have

$$\begin{aligned} &\sum_{i=1}^{N} \int_{E_t} \Big|\frac{\partial w_t}{\partial x_i}\Big|^2 \Big(\varepsilon + \Big|\frac{\partial u}{\partial x_i}\Big| + \Big|\frac{\partial v}{\partial x_i}\Big|\Big)^{p_i - 2} \\ &\leq C_2 \Big(\int_{E_t} w^{2^*}\Big)^{\frac{2}{2^*}} \sum_{i=1}^{N} \Bigg(\int_{E_t} h_i^N \Big(\Big|\frac{\partial u}{\partial x_i}\Big| + \Big|\frac{\partial v}{\partial x_i}\Big|\Big)^{(2\sigma_i - (p_i - 2))\frac{N}{2}}\Bigg)^{\frac{2}{N}} \\ &\quad + C_2 \Big(\int_{E_t} w^{2^*}\Big)^{\frac{2}{2^*}} \sum_{i=1}^{N} \Bigg(\int_{E_t} G_i^N \Big(\Big|\frac{\partial u}{\partial x_i}\Big| + \Big|\frac{\partial v}{\partial x_i}\Big|\Big)^{(2\sigma_i - (p_i - 2))\frac{N}{2}}\Bigg)^{\frac{2}{N}} \\ &\quad + M C_2 \Big(\int_{E_t} w^{2^*}\Big)^{\frac{2}{2^*}} |E_t|^{\frac{2}{N}}. \end{aligned}$$

Moreover, by (2.2), (2.3), (5.8), Young's and Hölder's inequalities, we have

$$\begin{aligned} \frac{1}{C_s^2} \Big(\int_{E_t} w^{2^*}\Big)^{\frac{2}{2^*}} &\leq \frac{C_2}{N\alpha} \Big(\int_{E_t} w^{2^*}\Big)^{\frac{2}{2^*}} \sum_{i=1}^{N} \Bigg(\int_{E_t} h_i^N \Big(\Big|\frac{\partial u}{\partial x_i}\Big| + \Big|\frac{\partial u}{\partial x_i}\Big|\Big)^{(2\sigma_i - (p_i - 2))\frac{N}{2}}\Bigg)^{\frac{2}{N}} \\ &\quad + \frac{C_2}{N\alpha} \Big(\int_{E_t} w^{2^*}\Big)^{\frac{2}{2^*}} \sum_{i=1}^{N} \Bigg(\int_{E_t} G_i^N \Big(\Big|\frac{\partial u}{\partial x_i}\Big| + \Big|\frac{\partial u}{\partial x_i}\Big|\Big)^{(2\sigma_i - (p_i - 2))\frac{N}{2}}\Bigg)^{\frac{2}{N}} \\ &\quad + \frac{C_2}{\alpha} \Big(\int_{E_t} w^{2^*}\Big)^{\frac{2}{2^*}} |E_t|^{\frac{2}{N}}, \end{aligned}$$

which gives

$$\begin{aligned} \frac{1}{C_s^2} &\leq \frac{C_2}{N\alpha} \sum_{i=1}^{N} \Bigg(\int_{E_t} h_i^N \Big(\Big|\frac{\partial u}{\partial x_i}\Big| + \Big|\frac{\partial v}{\partial x_i}\Big|\Big)^{(2\sigma_i - (p_i - 2))\frac{N}{2}}\Bigg)^{\frac{2}{N}} \\ &\quad + \frac{C_2}{N\alpha} \sum_{i=1}^{N} \Bigg(\int_{E_t} G_i^N \Big(\Big|\frac{\partial u}{\partial x_i}\Big| + \Big|\frac{\partial v}{\partial x_i}\Big|\Big)^{(2\sigma_i - (p_i - 2))\frac{N}{2}}\Bigg)^{\frac{2}{N}} + \frac{C_2}{\alpha} |E_t|^{\frac{2}{N}}. \end{aligned} \tag{5.10}$$

Since $\frac{N}{s_i} + \frac{(2\sigma_i-(p_i-2))N}{2p_i} \leq 1$, the right-hand side of (5.10) goes to zero for $t \to \sup w$. That gives a contradiction.

If $\sigma_i < \frac{p_i-2}{2}$, by (5.9) and Young's inequality, we have

$$\begin{aligned}\sum_{i=1}^{N}\int_{E_t}\Big|\frac{\partial w_t}{\partial x_i}\Big|^2 &\leq C_3\sum_{i=1}^{N}\int_{E_t}\Big|\frac{\partial w_t}{\partial x_i}\Big|^2\Big(\varepsilon+\Big|\frac{\partial u}{\partial x_i}\Big|+\Big|\frac{\partial v}{\partial x_i}\Big|\Big)^{p_i-2}\\ &\leq \frac{C_3\delta}{2}\sum_{i=1}^{N}\int_{E_t}\Big|\frac{\partial w_t}{\partial x_i}\Big|^2+\frac{C_3}{2\delta}\sum_{i=1}^{N}\int_{E_t}h_i^2\Big(\Big|\frac{\partial u}{\partial x_i}\Big|+\Big|\frac{\partial v}{\partial x_i}\Big|\Big)^{2\sigma_i}w_t^2\\ &\quad+\frac{C_3\delta}{2}\sum_{i=1}^{N}\int_{E_t}\Big|\frac{\partial w_t}{\partial x_i}\Big|^2+\frac{C_3}{2\delta}\sum_{i=1}^{N}\int_{E_t}G_i^2\Big(\Big|\frac{\partial u}{\partial x_i}\Big|+\Big|\frac{\partial v}{\partial x_i}\Big|\Big)^{2\sigma_i}w_t^2\\ &\quad+C_3\sum_{i=1}^{N}\int_{E_t}w^2.\end{aligned}$$

Choosing δ small enough, we get

$$\begin{aligned}\sum_{i=1}^{N}\int_{E_t}\Big|\frac{\partial w_t}{\partial x_i}\Big|^2 &\leq C_4\sum_{i=1}^{N}\int_{E_t}h_i^2\Big(\Big|\frac{\partial u}{\partial x_i}\Big|+\Big|\frac{\partial u}{\partial x_i}\Big|\Big)^{2\sigma_i}w_t^2\\ &\quad+C_4\sum_{i=1}^{N}\int_{E_t}G_i^2\Big(\Big|\frac{\partial u}{\partial x_i}\Big|+\Big|\frac{\partial v}{\partial x_i}\Big|\Big)^{2\sigma_i}w_t^2+C_4\sum_{i=1}^{N}\int_{E_t}w^2.\end{aligned} \tag{5.11}$$

We have $0 < w_t \leq w$ in E_t. Using the inequalities (2.2), (2.3), (5.11) and the Hölder inequality, we get

$$\begin{aligned}\frac{1}{C_s^2}\Big(\int_{E_t}w^{2^*}\Big)^{\frac{2}{2^*}} &\leq \prod_{i=1}^{N}\Big(\int_{E_t}\Big|\frac{\partial w_t}{\partial x_i}\Big|^2\Big)^{\frac{1}{N}} \leq \frac{1}{N}\sum_{i=1}^{N}\int_{E_t}\Big|\frac{\partial w_t}{\partial x_i}\Big|^2\\ &\leq \frac{C_4}{N}\Big(\int_{E_t}w^{2^*}\Big)^{\frac{2}{2^*}}\sum_{i=1}^{N}\Big(\int_{E_t}h_i^N\Big(\Big|\frac{\partial u}{\partial x_i}\Big|+\Big|\frac{\partial v}{\partial x_i}\Big|\Big)^{N\sigma_i}\Big)^{\frac{2}{N}}\\ &\quad+\frac{C_4}{N}\Big(\int_{E_t}w^{2^*}\Big)^{\frac{2}{2^*}}\sum_{i=1}^{N}\Big(\int_{E_t}G_i^N\Big(\Big|\frac{\partial u}{\partial x_i}\Big|+\Big|\frac{\partial v}{\partial x_i}\Big|\Big)^{N\sigma_i}\Big)^{\frac{2}{N}}\\ &\quad+C_4\Big(\int_{E_t}w^{2^*}\Big)^{\frac{2^*}{2}}|E_t|^{\frac{2}{N}}.\end{aligned}$$

Then, we have

$$\begin{aligned}\frac{1}{C_s^2} &\leq \frac{C_4}{N}\sum_{i=1}^{N}\Big(\int_{E_t}h_i^N\Big(\Big|\frac{\partial u}{\partial x_i}\Big|+\Big|\frac{\partial v}{\partial x_i}\Big|\Big)^{N\sigma_i}\Big)^{\frac{2}{N}}\\ &\quad+\frac{C_4}{N}\sum_{i=1}^{N}\Big(\int_{E_t}G_i^N\Big(\Big|\frac{\partial u}{\partial x_i}\Big|+\Big|\frac{\partial v}{\partial x_i}\Big|\Big)^{N\sigma_i}\Big)^{\frac{2}{N}}+C_4|E_t|^{\frac{2}{N}}.\end{aligned} \tag{5.12}$$

Since $\frac{N}{s_i} + \frac{N\sigma_i}{p_i} \leq 1$, the right-hand side of (5.12) goes to zero for $t \to \sup w$. That gives a contradiction. $\square$

References

[1] Y. Akdim, E. Azroul, A. Benkirane, *Existence results for quasilinear degenerated equations via strong convergence of truncations*, Revista Matemática Complutense **17** (2004), no. 2, 359–379.

[2] Y. Akdim, A. Benkirane, M. El Moumni, *Existence results for nonlinear elliptic problems with lower order terms*, International Journal of Evolution Equations **8** (2013), no. 4, 257–276.

[3] A. Alvino, M. F. Betta, A. Mercaldo, *Comparison principle for some classes of nonlinear elliptic equations*, Journal of Differential Equations **249** (2010), no. 12, 3279–3290.

[4] S. Antontsev, M. Chipot, *Anisotropic equations: uniqueness and existence results*, Differential and Integral Equations **21** (2008), no. 5–6, 401–419.

[5] M. Bendahmane, K. H. Karlsen, *Anisotropic nonlinear elliptic systems with measure data and anisotropic harmonic maps into spheres*, Electronic Journal of Differential Equations **2006** (2006), no. 46, 30 pages.

[6] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, J. Vázquez, *An L^1-theory of existence and uniqueness of nonlinear elliptic equations*, Annali della Scuola Normale Superiore di Pisa. Classe di Scienze. Serie IV **22** (1995), no. 2, 241–273.

[7] L. Boccardo, T. Gallouët, *Strongly nonlinear elliptic equations having natural growth terms and L^1 data*, Nonlinear Analysis: Theory, Methods & Applications **19** (1992), no. 6, 573–579.

[8] L. Boccardo, T. Gallouët, P. Marcellini, *Anisotropic equations in L^1*, Differential and Integral Equations **9** (1996), no. 1, 209–212.

[9] L. Boccardo, T. Gallouët, L. Orsina, *Existence and nonexistence of solutions for some nonlinear elliptic equations*, Journal d'Analyse Mathématique **73** (1997), 203–223.

[10] L. Boccardo, F. Murat, J.-P. Puel, *Existence of bounded solutions for nonlinear elliptic unilateral problems*, Annali di Matematica Pura ed Applicata **152** (1988), 183–196.

[11] G. Bottaro, M. E. Marina, *Problemi di Dirichlet per equazioni ellittiche di tipo variazionale su insiemi non limitati*, Bollettino dell'Unione Matematica Italiana **8** (1973), 46–56.

[12] A. Di Castro, *Existence and regularity results for anisotropic elliptic problems*, Advanced Nonlinear Studies **9** (2009), no. 2, 367–393.

[13] A. Di Castro, *Anisotropic elliptic problems with natural growth terms*, Manuscripta Mathematica **135** (2011), no. 3–4, 521–543.

[14] R. Di Nardo, F. Feo, *Existence and uniqueness for nonlinear anisotropic elliptic equations*, Archiv der Mathematik **102** (2014), no. 2, 141–153.

[15] I. Fragala, F. Gazzola, B. Kawohl, *Existence and nonexistence results for anisotropic quasilinear elliptic equations*, Annales de l'Institut Henri Poincaré Analyse Non Linéaire **21** (2004), no. 5, 715–734.

[16] O. Guibé, A. Mercaldo, *Uniqueness results for noncoercive nonlinear elliptic equations with two lower order terms*, Communications on Pure and Applied Analysis **7** (2008), no. 1, 163–192.

[17] J. Leray, J.-L. Lions, *Quelques résulatats de Višik sur les problèmes elliptiques nonlinéaires par les méthodes de Minty-Browder*, Bulletin de la Société Mathématique de France **93** (1965), 97–107.

[18] F.-Q. Li, *Anisotropic elliptic equations in L^m*, Journal of Convex Analysis **8** (2001), no. 2, 417–422.

[19] V.-M. Monetti, L. Randazzo, *Existence results for nonlinear elliptic equations with p-growth in the gradient*, Ricerche di Matematica **49** (2000), no. 1, 163–181.

[20] M. Troisi, *Teoremi di inclusione per spazi di Sobolev non isotropi*, Ricerche di Matematica **18** (1969), 3–24.