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APPROXIMATIONS OF SOLUTIONS TO NEUTRAL RETARDED INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. This paper deals with a Sobolev type retarded integro-differential equation. We prove existence, uniqueness and convergence of each integral approximate solution using analytic semigroup theory and fixed point method. Then we consider Faedo–Galerkin approximation of solutions and prove some convergence results. We also give some examples to illustrate the applications of the abstract results.

Keywords: Analytic semigroup, delay, integro-differential equation, neutral differential equation.

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1 Introduction

In this paper we study the existence, uniqueness and approximation of mild solutions of the following neutral integro-differential equation in a separable Hilbert space H:

$$\begin{aligned} &\frac{\mathrm{d}(u(t)+g(t,u(t)))}{\mathrm{d}t} + Au(t) = Bu(t) + Cu(t-\tau) + \int_{-\tau}^{0} a(\theta) Lu(t+\theta)\,\mathrm{d}\theta, \\ &\qquad\qquad\qquad\qquad 0 < t \le T < \infty, \quad \tau > 0, \\ &u(t) = h(t), \quad t \in [-\tau, 0]. \end{aligned} \tag{1.1}$$

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Here u is a function from $[-\tau,\infty)$ into the space H, $h\colon[-\tau,0]\to H$ is a given function and $a\in L^p_{\text{loc}}(-\tau,0)$. For each $t\geq 0$, $u_t\colon[-\tau,0]\to H$ is defined by $u_t(\theta)=u(t+\theta)$, $\theta\in[-\tau,0]$, and the operator $A\colon D(A)\subseteq H\to H$ is a linear operator. The operators $B\colon D(B)\subseteq H\to H$, $C\colon D(C)\subseteq H\to H$ and $L\colon D(L)\subseteq H\to H$ are non-linear continuous operators.

For $t\in[0,T]$, we shall use the notation $\mathcal{C}_t:=C([-\tau,t];H)$ for the Banach space of all continuous functions from $[-\tau,t]$ into H endowed with the supremum norm

$$\|\psi\|_t:=\sup_{-\tau\leq\eta\leq t}\|\psi(\eta)\|.$$

The existence, uniqueness and regularity of solutions of (1.1) under different conditions have been studied by Di Blasio *et al.* [8] and Jeong *et al.* [12]. The fundamental works on the existence, uniqueness and stability of various types of solutions of functional differential equations are Bahuguna [1, 2], Balachandran and Chandrasekaran [5], Lin and Liu [13]. The related results for the approximation of solutions may be found in Bahuguna, Srivastava and Singh [4] and Bahuguna and Shukla [3].

Segal [17] and Murakami [15] studied the existence, uniqueness and finite-time blow-up of solutions for the following equation

$$\begin{aligned} u'(t)+Au(t)&=g(u(t)),\quad t\geq 0,\\ u(0)&=\phi. \end{aligned}\tag{1.2}$$

Bazley [6, 7] studied the following semilinear wave equation

$$\begin{aligned} u''(t)+Au(t)&=g(u(t)),\quad t\geq 0,\\ u(0)=\phi,&\quad u'(0)=\psi, \end{aligned}\tag{1.3}$$

and proved the uniform convergence of approximations of solutions to (1.3) using the existence results of Heinz and von Wahl [11]. Goethel [10] proved the convergence of approximations of solutions to (1.2). Goethel [10] assumed g to be defined on the whole of H. From the methodology of Bazley [6, 7], Miletta [14] proved the convergence of approximations to solutions of (1.2). In our paper, we use the methods of Miletta [14] and Bahuguna *et al.* [3, 4] with suitable modifications to prove the convergence of finite dimensional approximations of the solutions to (1.1). We use the Banach contraction principle to prove our first theorem.

2 Preliminaries and assumptions

The existence of a solution to (1.1) is closely associated with the existence of a function $u\in\mathcal{C}_{\tilde{T}}$, $0<\tilde{T}\leq T$, satisfying

$$u(t)=\begin{cases} h(t), & t\in[-\tau,0],\\ e^{-tA}\big(h(0)+g(0,h(0))\big)-g(t,u(t))+\displaystyle\int_0^t Ae^{-(t-s)A}g(s,u(s))\,\mathrm{d}s & \\ \quad+\displaystyle\int_0^t e^{-(t-s)A}\left[Bu(s)+Cu(s-\tau)+\int_{-\tau}^0 a(\theta)Lu(s+\theta)\,\mathrm{d}\theta\right]\mathrm{d}s, & t\in[0,\tilde{T}],\end{cases}$$

and such a function u is called a *mild solution* of (1.1) on $[-\tau,\tilde{T}]$. A function $u\in\mathcal{C}_{\tilde{T}}$ is called a *classical solution* of (1.1) on $[-\tau,\tilde{T}]$, if $u\in C^1((0,\tilde{T}];H)$ and u satisfies (1.1) on $[-\tau,\tilde{T}]$.

We assume in (1.1) that the linear operator A satisfies the following conditions.

(H1) A is a closed, positive definite, self-adjoint linear operator from the domain $D(A) \subseteq H$ into H such that $D(A)$ is dense in H; A has the pure point spectrum

$$0 < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \dots$$

and a corresponding complete orthonormal system of eigenfunctions $\{\phi_i\}$, *i.e.*,

$$A\phi_i = \lambda_i \phi_i \quad \text{and} \quad (\phi_i, \phi_j) = \delta_{ij},$$

where $\delta_{ij} = 1$ if $i = j$ and zero otherwise.

If **(H1)** is satisfied then $-A$ is the infinitesimal generator of an analytic semigroup $\{e^{-tA} : t \geq 0\}$ in H (cf. [16, pp. 60–69]). It follows that the fractional powers A^α of A for $0 \leq \alpha \leq 1$ are well defined from $D(A^\alpha) \subseteq H$ into H (cf. [16, pp. 69–75]). Hence, for convenience, we suppose that

$$\|e^{-tA}\| \leq M \quad \text{for all} \quad t \geq 0$$

and $0 \in \rho(-A)$, where $\rho(-A)$ is the resolvent set of $-A$.

$D(A^\alpha)$ is a Banach space endowed with the norm $\|x\|_\alpha = \|A^\alpha x\|$.

For $t \in [0, T]$, we set $\mathcal{C}_t^\alpha := C([-\tau, t]; D(A^\alpha))$ and endow this space with the norm

$$\|\psi\|_{t,\alpha} := \sup_{-\tau \leq \nu \leq t} \|\psi(\nu)\|_\alpha.$$

Further, we make the following assumptions.

(H2) $h \in \mathcal{C}_0^\alpha$ and h is locally Hölder continuous on $[-\tau, 0]$.

(H3) We shall assume that the map $B\colon D(A^\alpha) \to H$ satisfies the following Lipschitz condition on balls in $D(A^\alpha)$: for each $\eta > 0$ and some $0 < \alpha < 1$ there exists a constant $K_1(\eta)$ such that

(i) $\|B(\psi)\| \leq K_1(\eta)$ for $\psi \in D(A^\alpha)$ with $\|A^\alpha \psi\| \leq \eta$;

(ii) $\|B(\psi_1) - B(\psi_2)\| \leq K_1(\eta)\|A^\alpha(\psi_1 - \psi_2)\|$ for $\psi_1, \psi_2 \in D(A^\alpha)$ with $\|A^\alpha \psi_i\| \leq \eta$ for $i = 1, 2$.

(H4) The map $C\colon D(A^\alpha) \to H$ satisfies the following Lipschitz condition on balls in $D(A^\alpha)$: for each $\eta > 0$ and some $0 < \alpha < 1$ there exists a constant $K_2(\eta)$ such that

(iii) $\|C(\psi)\| \leq K_2(\eta)$ for $\psi \in D(A^\alpha)$ with $\|A^\alpha \psi\| \leq \eta$;

(iv) $\|C(\psi_1) - C(\psi_2)\| \leq K_2(\eta)\|A^\alpha(\psi_1 - \psi_2)\|$ for $\psi_1, \psi_2 \in D(A^\alpha)$ with $\|A^\alpha \psi_i\| \leq \eta$ for $i = 1, 2$.

(H5) The map $L\colon D(A^\alpha) \to H$ satisfies the following Lipschitz condition on balls in $D(A^\alpha)$: for each $\eta > 0$ and some $0 < \alpha < 1$ there exists a constant $K_3(\eta)$ such that

(v) $\|L(\psi)\| \leq K_3(\eta)$ for $\psi \in D(A^\alpha)$ with $\|A^\alpha \psi\| \leq \eta$;

(vi) $\|L(\psi_1) - L(\psi_2)\| \leq K_3(\eta)\|A^\alpha(\psi_1 - \psi_2)\|$ for $\psi_1, \psi_2 \in D(A^\alpha)$ with $\|A^\alpha \psi_i\| \leq \eta$ for $i = 1, 2$.

(H6) $a \in L^p_{\text{loc}}(-\tau, 0)$ for some $1 < p < \infty$ and $a_T = \int_{-\tau}^0 |a(\theta)| \, \mathrm{d}\theta$.

(Hg) The map $g\colon [0,T] \times D(A) \to H$ satisfies the following Lipschitz condition on balls in $D(A^\alpha)$: for each $\eta > 0$ and some $0 < \alpha < 1$ there exists a constant $L_g(\eta)$ such that

(vii) $\|g(\psi)\| \le L_g(\eta)$ for $\psi \in D(A^\alpha)$ with $\|A^\alpha \psi\| \le \eta$;

(viii) $\|g(t_1,\psi_1) - g(t_2,\psi_2)\| \le L_g(\eta)\big(\|t_1 - t_2\|^\gamma + \|A^\alpha(\psi_1 - \psi_2)\|\big)$ for $\psi_1, \psi_2 \in D(A^\alpha)$ with $\|A^\alpha \psi_i\| \le \eta$ for $i = 1, 2$.

3 Approximate solutions and convergence

Let H_n denote the finite dimensional subspace of H spanned by $\{\phi_0, \phi_1, \cdots, \phi_n\}$ and let $P^n\colon H \to H_n$ be the corresponding projection operator for $n = 0, 1, 2, \cdots$. Let $0 < T_0 \le T$ be such that

$$\sup_{0 \le t \le T_0} \|(e^{-tA} - I)A^\alpha h(0)\| \le \frac{R}{2}, \tag{3.1}$$

where $R > 0$ is a fixed quantity.

Let us define

$$\bar{h}(t) = \begin{cases} h(t), & \text{if } t \in [-\tau, 0], \\ h(0), & \text{if } t \in [0, T]. \end{cases}$$

We set

$$T_0 < \min\left[\left\{\frac{R}{2}(1-\alpha)(K(\eta_0)C_\alpha)^{-1}\right\}^{\frac{1}{1-\alpha}}, \left\{\frac{1}{2}(1-\alpha)(K(\eta_0)C_\alpha)^{-1}\right\}^{\frac{1}{1-\alpha}}\right], \tag{3.2}$$

where

$$K(\eta_0) = \big[K_1(\eta_0) + K_2(\eta_0) + K_3(\eta_0)a_T\big] \tag{3.3}$$

and C_α is a positive constant such that $\|A^\alpha e^{-tA}\| \le C_\alpha t^{-\alpha}$ for $t > 0$. We define $B_n\colon H \to H$ by

$$B_n x = BP^n x, \quad x \in H.$$

Similarly, C_n and L_n are given by

$$C_n x = CP^n x, \quad x \in H, \qquad L_n x = LP^n x, \quad x \in H.$$

Let $A^\alpha\colon \mathcal{C}_t^\alpha \to \mathcal{C}_t$ be given by $(A^\alpha \psi)(s) = A^\alpha(\psi(s))$, $s \in [-\tau, t]$, $t \in [0, T]$. We define the map F_n on $B_R(\mathcal{C}_{T_0}^\alpha, \bar{h})$ as follows

$$(F_n u)(t) = \begin{cases} h(t), & t \in [-\tau, 0], \\ e^{-tA}\big(h(0) + g_n(0, h(0))\big) - g_n(t, u(t)) + \displaystyle\int_0^t Ae^{-(t-s)A} g_n(s, u)\, \mathrm{d}s & \\ \quad + \displaystyle\int_0^t e^{-(t-s)A}\left[B_n u(s) + C_n u(s-\tau) + \int_{-\tau}^0 a(\theta) L_n u(s+\theta)\, \mathrm{d}\theta\right] \mathrm{d}s, & t \in [0, T_0], \end{cases}$$

for $u \in B_R(\mathcal{C}_{T_0}^\alpha, \bar{h})$.

Theorem 1 *Suppose that the conditions* **(H1)–(Hg)** *are satisfied and $h(t) \in D(A)$ for all $t \in [-\tau, 0]$. Then there exists a unique $u_n \in B_R(\mathcal{C}^\alpha_{T_0}, \bar{h})$ such that $F_n u_n = u_n$ for each $n = 0, 1, 2, \cdots$,* i.e., *u_n satisfies the approximate integral equation*

$$u_n(t) = \begin{cases} h(t), & t \in [-\tau, 0], \\ e^{-tA}\big(h(0) + g_n(0, h(0))\big) - g_n(t, u_n(t)) & \\ \quad + \displaystyle\int_0^t A e^{-(t-s)A} g_n(s, u_n)\, \mathrm{d}s + \int_0^t e^{-(t-s)A} \Big[B_n u_n(s) & \\ \quad\quad + C_n u_n(s-\tau) + \displaystyle\int_{-\tau}^0 a(\theta) L_n u_n(s+\theta)\, \mathrm{d}\theta \Big] \mathrm{d}s, & t \in [0, T_0]. \end{cases} \tag{3.4}$$

Proof. First, we show that $F_n \colon B_R(\mathcal{C}^\alpha_{T_0}, \bar{h}) \to B_R(\mathcal{C}^\alpha_{T_0}, \bar{h})$. For this first we need to show that the map $t \mapsto (F_n u)(t)$ is continuous from $[-\tau, T_0]$ into $D(A^\alpha)$ with respect to the norm $\|\cdot\|_\alpha$. For any $u \in B_R(\mathcal{C}^\alpha_{T_0}, \bar{h})$ and $t_1, t_2 \in [-\tau, 0]$, we have

$$(F_n u)(t_1) - (F_n u)(t_2) = h(t_1) - h(t_2). \tag{3.5}$$

Now, for $t_1, t_2 \in (0, T_0]$ with $t_1 < t_2$, we have

$$\begin{aligned} &\|(F_n u)(t_2) - (F_n u)(t_1)\|_\alpha \\ &\quad \le \|(e^{-t_2 A} - e^{-t_1 A})(h(0) + g(0, h(0)))\|_\alpha \\ &\quad + \|A^{\alpha-\beta}\| \, \|A^\beta g_n(t_2, u) - A^\beta g_n(t_1, u)\| \\ &\quad + \int_0^{t_1} \|(e^{-(t_2-t_1)A} - I) A^{1+\alpha-\beta} e^{-(t_1-s)A}\| \, \|A^\beta g_n(s, u)\| \, \mathrm{d}s \\ &\quad + \int_{t_1}^{t_2} \|e^{-(t_2-s)A} A^{1+\alpha-\beta}\| \, \|A^\beta g_n(s, u)\| \, \mathrm{d}s \\ &\quad + \int_0^{t_1} \|(e^{-(t_2-s)A} - e^{-(t_1-s)A}) A^\alpha\| \\ &\qquad \times \Big[\|B_n u(s)\| + \|C_n u(s-\tau)\| + \int_{-\tau}^0 |a(\theta)| \|L_n u(s+\theta)\| \, \mathrm{d}\theta \Big] \mathrm{d}s \\ &\quad + \int_{t_1}^{t_2} \|(e^{-(t_2-s)A}) A^\alpha\| \Big[\|B_n u(s)\| + \|C_n u(s-\tau)\| \\ &\qquad + \int_{-\tau}^0 |a(\theta)| \|L_n u(s+\theta)\| \, \mathrm{d}\theta \Big] \mathrm{d}s. \end{aligned} \tag{3.6}$$

Using **(Hg)** we obtain

$$\begin{aligned} \|A^\beta g_n(t_2, u) - A^\beta g_n(t_1, u)\| &\le L_g\big(h^\gamma + \|P^n u(t_2) - P^n u(t_1)\|_\alpha\big) \\ &\le L_g\big(h^\gamma + \|u(t_2) - u(t_1)\|_\alpha\big) \end{aligned} \tag{3.7}$$

and

$$\int_{t_1}^{t_2} \|e^{-(t_2-s)A} A^{1+\alpha-\beta}\| \, \|A^\beta g_n(s, u)\| \, \mathrm{d}s \le \frac{(L_g \tilde{R} + B) C_{1+\alpha-\beta} h^{\beta-\alpha}}{\beta - \alpha}, \tag{3.8}$$

since

$$\begin{aligned}\|A^\beta g_n(s,u)\| &\le \|A^\beta g_n(s,u) - A^\beta g(s,h(0))\| + \|A^\beta g(s,h(0))\| \\ &\le L_g\|P^n u(s) - h(0)\|_\alpha + B \le L_g\tilde{R} + B.\end{aligned} \tag{3.9}$$

Part (d) of [16, Theorem 2.6.13] implies that for $0 < \vartheta \le 1$ and $x \in D(A^\vartheta)$,

$$\|(e^{-tA} - I)x\| \le C'_\vartheta t^\vartheta \|x\|_\vartheta. \tag{3.10}$$

Let ϑ be a real number with $0 < \vartheta < \min\{1-\alpha, \beta-\alpha\}$. Then $A^\alpha y \in D(A^\vartheta)$ for any $y \in D(A^{\alpha+\vartheta})$. For all $t, s \in [0,T]$, $t \ge s$ and we get the following inequalities:

$$\|(e^{-(t_2-t_1)A} - I)A^\alpha e^{-t_1A}\| \le C'_\vartheta (t_2-t_1)^\vartheta \|A^{\alpha+\vartheta} e^{-t_1A}\| \le \frac{\tilde{C}(t_2-t_1)^\vartheta}{t_1^{\alpha+\vartheta}}, \tag{3.11}$$

$$\|(e^{-(t_2-t_1)A} - I)A^\alpha e^{-(t_1-s)A}\| \le \frac{\tilde{C}(t_2-t_1)^\vartheta}{(t_1-s)^{\alpha+\vartheta}}, \tag{3.12}$$

$$\|(e^{-(t_2-t_1)A} - I)A^{1+\alpha-\beta} e^{-(t_1-s)A}\| \le \frac{\tilde{C}(t_2-t_1)^\vartheta}{t_1^{1+\alpha+\vartheta-\beta}}, \tag{3.13}$$

where $\tilde{C} = C'_\vartheta \max\{C_{\alpha+\vartheta}, C_{1+\alpha+\vartheta-\beta}\}$. Using the estimates (3.9), (3.12) and (3.13), we get

$$\begin{aligned}\int_0^{t_1} &\|(e^{-(t_2-t_1)A} - I)A^{1+\alpha-\beta} e^{-(t_1-s)A}\| \, \|A^\beta g_n(s,u)\| \, \mathrm{d}s \\ &\le \tilde{C}(t_2-t_1)^\vartheta (L_g\tilde{R} + B) \frac{T_0^{\beta-(\alpha+\vartheta)}}{\beta - (\alpha+\vartheta)}.\end{aligned} \tag{3.14}$$

Part (d) of [16, Theorem 2.6.13] states that for $0 < \beta \le 1$ and $x \in D(A^\beta)$,

$$\|(e^{-tA} - I)x\| \le C_\beta t^\beta \|A^\beta x\|.$$

Hence, if $0 < \beta < 1$ is such that $0 < \alpha + \beta < 1$, then $A^\alpha y \in D(A^\beta)$. Therefore, for $t, s \in (0, T_0]$, we have

$$\|(e^{-tA} - I)A^\alpha e^{-sA} x\| \le C_\beta t^\beta \|A^{\alpha+\beta} e^{-sA} x\| \le C_\beta C_{\alpha+\beta} t^\beta s^{-(\alpha+\beta)} \|x\|. \tag{3.15}$$

We use inequality (3.15) to obtain

$$\begin{aligned}\int_0^{t_1} &\|(e^{-(t_2-s)A} - e^{-(t_1-s)A})A^\alpha\| \Big[\|B_n u(s)\| + \|C_n u(s-\tau)\| \\ &\qquad + \int_{-\tau}^0 |a(\theta)| \|L_n u(s+\theta)\| \, \mathrm{d}\theta\Big] \mathrm{d}s \\ \le \int_0^{t_1} &\|(e^{-(t_2-t_1)A} - I)e^{-(t_1-s)A} A^\alpha\| \Big[\|B_n u(s)\| + \|C_n u(s-\tau)\| \\ &\qquad + \int_{-\tau}^0 |a(\theta)| \|L_n u(s+\theta)\| \, \mathrm{d}\theta\Big] \mathrm{d}s \\ \le C_{\alpha,\beta}&(t_2-t_1)^\beta,\end{aligned} \tag{3.16}$$

where

$$C_{\alpha,\beta} = C_\beta C_{\alpha+\beta} K(\eta_0) \frac{T_0^{1-(\alpha+\beta)}}{[1-(\alpha+\beta)]},$$

$K(\eta_0)$ is given by (3.3) and $\eta_0 = R + \|h\|_{0,\alpha}$. We calculate the last integral on the right-hand side of (3.6) as follows. We have

$$\begin{aligned}\int_{t_1}^{t_2} \|e^{-(t_2-s)A}A^\alpha\| \left[\|B_n u(s)\| + \|C_n u(s-\tau)\| + \int_{-\tau}^{0} |a(\theta)| \|L_n u(s+\theta)\| \,\mathrm{d}\theta\right] \mathrm{d}s \\ \leq C_\alpha K(\eta_0) \frac{(t_2-t_1)^{1-\alpha}}{(1-\alpha)}.\end{aligned} \tag{3.17}$$

Hence, from (3.5), (3.7), (3.8), (3.9), (3.14), (3.16) and (3.17), the map $t \mapsto (F_n u)(t)$ is continuous from $[-\tau, T_0]$ into $D(A^\alpha)$ with respect to the norm $\|\cdot\|_\alpha$.

Now, for $t \in [-\tau, 0]$, $(F_n u)(t) - \bar{h}(t) = 0$. For $t \in (0, T_0]$, we have

$$\begin{aligned}&\|(F_n u)(t) - \bar{h}(t)\|_\alpha \\ &\quad\leq \|(e^{-tA} - I)A^\alpha(h(0) + g_n(0, h(0)))\| + \|A^{\alpha-\beta}\| \, \|A^\beta g_n(0, h(0)) - A^\beta g_n(t, u)\| \\ &\qquad + \int_0^t \|A^{1+\alpha-\beta} e^{-(t-s)A}\| \, \|A^\beta g_n(s, u)\| \,\mathrm{d}s \\ &\qquad + \int_0^t \|e^{-(t-s)A}A^\alpha\| \left[\|B_n u(s)\| + \|C_n u(s-\tau)\| + \int_{-\tau}^0 |a(\theta)| \|L_n u(s+\theta)\| \,\mathrm{d}\theta\right] \mathrm{d}s \\ &\quad\leq (1-\mu)\frac{R}{3} + \|A^{\alpha-\beta}\| L_g \{T^\gamma + \|u(t) - \phi\|_\alpha\} \\ &\qquad + C_{1+\alpha-\beta}(L_g \tilde{R} + B)\frac{T^{\beta-\alpha}}{\beta-\alpha} + C_\alpha K(\eta_0)\frac{T^{1-\alpha}}{1-\alpha} \\ &\quad\leq (1-\mu)\frac{R}{3} + (1-\mu)\frac{R}{6} + C_\alpha K(\eta_0)\frac{T_0^{1-\alpha}}{1-\alpha} \\ &\quad\leq \frac{R}{2} + C_\alpha K(\eta_0)\frac{T_0^{1-\alpha}}{1-\alpha}.\end{aligned}$$

Hence $\|F_n u - \bar{h}\|_{T_0,\alpha} \leq R$. Thus $F_n \colon B_R(\mathcal{C}^\alpha_{T_0}, \bar{h}) \to B_R(\mathcal{C}^\alpha_{T_0}, \bar{h})$.

Now, for any $u, v \in B_R(\mathcal{C}^\alpha_{T_0}, \bar{h})$ and $t \in [-\tau, 0]$, we have $F_n u(t) - F_n v(t) = 0$. Moreover

$$\|A^\beta g_n(t, u) - A^\beta g_n(t, v)\| \leq L_g \|u(t) - v(t)\|_\alpha \leq L_g \|u - v\|_{T_0,\alpha}. \tag{3.18}$$

Thus, for $t \in (0, T_0]$ and $u, v \in B_R(\mathcal{C}^\alpha_{T_0}, \bar{h})$, we have

$$\begin{aligned}&\|F_n u(t) - F_n v(t)\|_\alpha \\ &\quad\leq \|A^{\alpha-\beta}\| \, \|A^\beta g_n(t, u) - A^\beta g_n(t, v)\|_\alpha \\ &\qquad + \int_0^t \|A^{1+\alpha-\beta} e^{-(t-s)A}\| \, \|A^\beta g_n(s, u) - A^\beta g_n(s, v)\| \,\mathrm{d}s \\ &\qquad + \int_0^t \|e^{-(t-s)A}A^\alpha\| \Big[\|B_n u(s) - B_n v(s)\| + \|C_n u(s-\tau) - C_n v(s-\tau)\| \\ &\qquad + \int_{-\tau}^0 |a(\theta)| \|L_n u(s+\theta) - L_n v(s+\theta)\| \,\mathrm{d}\theta\Big] \mathrm{d}s \\ &\quad\leq \left(\|A^{\alpha-\beta}\| L + C_{1+\alpha-\beta} L \frac{T_0^{\beta-\alpha}}{\beta-\alpha}\right) \|u - v\|_{T_0,\alpha}\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t C_\alpha(t-s)^{-\alpha}\bigg[K_1(\eta_0)\|u(s)-v(s)\|_\alpha + K_2(\eta_0)\|u(s-\tau)-v(s-\tau)\|_\alpha \\
&+\int_{-\tau}^0 |a(\theta)|K_3(\eta_0)\|u(s+\theta)-v(s+\theta)\|_\alpha \,\mathrm{d}\theta\bigg]\mathrm{d}s \\
\le& \left(\|A^{\alpha-\beta}\|L_g + C_{1+\alpha-\beta}L\frac{T_0^{\beta-\alpha}}{\beta-\alpha}\right)\|u-v\|_{T_0,\alpha} \\
&+\int_0^t C_\alpha(t-s)^{-\alpha}K(\eta_0)\|u-v\|_{T_0,\alpha}\,\mathrm{d}s \\
\le& \left(\|A^{\alpha-\beta}\|L_g + C_{1+\alpha-\beta}L\frac{T_0^{\beta-\alpha}}{\beta-\alpha}\right)\|u-v\|_{T_0,\alpha} + \frac{1}{2}\|u-v\|_{T_0,\alpha}.
\end{aligned}
$$

Taking the supremum on t over $[-\tau, T_0]$, we get

$$
\|F_n u - F_n v\|_{T_0,\alpha} \le \left(\|A^{\alpha-\beta}\|L_g + C_{1+\alpha-\beta}L\frac{T_0^{\beta-\alpha}}{\beta-\alpha} + \frac{1}{2}\right)\|u-v\|_{T_0,\alpha}.
$$

Hence, by the definition of T_0, there exists a unique $u_n \in B_R(\mathcal{C}_{T_0}^\alpha, \bar{h})$ such that $F_n u_n = u_n$, which satisfies the approximate integral equation (3.4). This completes the proof of Theorem 1. □

Corollary 1 *If all the hypotheses of Theorem* 1 *are satisfied, then* $u_n(t) \in D(A^\beta)$ *for all* $t \in [-\tau, T_0]$*, where* $0 \le \beta < 1$.

Proof. From Theorem 1 there exists a unique $u_n \in B_R(\mathcal{C}_{T_0}^\alpha, \bar{h})$ satisfying (3.4). From [16, Theorem 1.2.4] we have that $e^{-tA}x \in D(A)$ for $x \in D(A)$. Also from Part (a) of [16, Theorem 2.6.13] we have $e^{-tA}\colon H \to D(A^\beta)$ for $t > 0$ and $0 \le \beta < 1$. Hölder continuity of u_n follows from similar arguments to those used in (3.16) and (3.17). Thus, **(Hg)** implies that the map $t \mapsto A^\beta g(t, u_n(t))$ is Hölder continuous on $[0, T]$ with the exponent ρ. It follows from [16, Theorem 4.3.2] that

$$
\int_0^t e^{-(t-s)A}A^\beta g_n(s, u_n)\,\mathrm{d}s \in D(A),
$$

and for $0 < t < T$ we have

$$
\int_0^t e^{-(t-s)A} f(s)\,\mathrm{d}s \in D(A).
$$

Since $D(A) \subseteq D(A^\beta)$ for $0 \le \beta \le 1$, the result of Corollary 1 thus follows. □

Corollary 2 *If* $h(0) \in D(A^\alpha)$*, where* $0 < \alpha < 1$ *and* $t_0 \in (0, T_0]$*, then there exists a constant* M_{t_0}*, independent of* n*, such that*

$$
\|A^\beta u_n(t)\| \le M_{t_0}
$$

for all $t_0 \le t \le T_0$ *and* $0 \le \beta < 1$*. Furthermore, if* $h(t) \in D(A)$ *for all* $t \in [-\tau, 0]$*, then there exists a constant* M_0*, independent of* n*, such that*

$$
\|A^\beta u_n(t)\| \le M_0
$$

for all $-\tau \le t \le T_0$ *and* $0 \le \beta < 1$.

Proof. For any $t_0 \in (0, T_0]$, we have

$$\|u_n(t)\|_\beta \leq C_\beta t_0^{-\beta}\big(\|h(0)\| + \|g_n(0, h(0))\|\big) + \|A^{\beta-\alpha}\|(L_g\tilde{R} + B) + C_{1+\beta-\alpha}(L_g\tilde{R} + B)\frac{T^{1-\beta}}{1-\beta} + C_\beta K(\eta_0)\frac{T_0^{1-\beta}}{1-\beta} \leq M_{t_0}.$$

Now, as $h(t) \in D(A)$ for all $t \in [-\tau, 0]$, $h(t) \in D(A^\beta)$ for all $t \in [-\tau, 0]$, and so for any $t \in [-\tau, 0]$ we have

$$\|u_n(t)\|_\beta = \|A^\beta h(t)\| \leq \|h\|_{0,\beta} \quad \text{for all } t \in [-\tau, 0].$$

Now again, for any $t \in (0, T_0]$, we have

$$\|u_n(t)\|_\beta \leq M\big(\|h\| + \|g_n(0, h(0))\|\big)_{0,\beta} + \|A^{\beta-\alpha}\|(L_g\tilde{R} + B) + C_{1+\beta-\alpha}(L\tilde{R} + B)\frac{T^{1-\beta}}{1-\beta} + C_\beta K(\eta_0)\frac{T_0^{1-\beta}}{1-\beta}.$$

This completes the proof of Corollary 2. □

Theorem 2 *Suppose that the conditions* **(H1)–(Hg)** *are satisfied and $h(t) \in D(A)$ for all $t \in [-\tau, 0]$. Then the sequence $\{u_n\} \subseteq \mathcal{C}_{T_0}^\alpha$ is a Cauchy sequence and therefore converges to a function $u \in \mathcal{C}_{T_0}^\alpha$.*

Proof. For $n \geq m \geq n_0$, where n_0 is large enough, $n, m, n_0 \in \mathbb{N}$, $t \in [-\tau, 0]$, we have

$$\|u_n(t) - u_m(t)\|_\alpha = \|h(t) - h(t)\|_\alpha = 0. \tag{3.19}$$

For $t \in (0, T_0]$ and n, m and n_0 as above, we have

$$\begin{aligned}
&\|u_n(t) - u_m(t)\|_\alpha \\
&\quad \leq \|e^{-tA}A^\alpha(g_n(0,\tilde{\phi}) - g_m(0,\tilde{\phi}))\| + \|A^{\alpha-\beta}\|\,\|A^\beta g_n(t,u_n) - A^\beta g_m(t,u_m)\| \\
&\quad + \int_0^t \|A^{1+\alpha-\beta}e^{-(t-s)A}\|\,\|A^\beta g_n(s,u_n) - A^\beta g_m(s,u_m)\|\,\mathrm{d}s \\
&\quad + \int_0^t \|e^{-(t-s)A}A^\alpha\|\Big[\|B_n u_n(s) - B_m u_m(s)\| \\
&\quad + \|C_n u_n(s-\tau) - C_m u_m(s-\tau)\| \\
&\quad + \int_{-\tau}^0 |a(\theta)|\|L_n u_n(s+\theta) - L_m u_m(s+\theta)\|\,\mathrm{d}\theta\Big]\mathrm{d}s.
\end{aligned}$$

For $0 < t_0' < t_0$, we have

$$\begin{aligned}
&\|u_n(t) - u_m(t)\|_\alpha \\
&\quad \leq \|e^{-tA}A^\alpha(g_n(0,\tilde{\phi}) - g_m(0,\tilde{\phi}))\| + \|A^{\alpha-\beta}\|\,\|A^\beta g_n(t,u_n) - A^\beta g_m(t,u_m)\| \\
&\quad + \left(\int_0^{t_0'} + \int_{t_0'}^t\right)\|A^{1+\alpha-\beta}e^{-(t-s)A}\|\,\|A^\beta g_n(s,u_n) - A^\beta g_m(s,u_m)\|\mathrm{d}s \\
&\quad + \left(\int_0^{t_0'} + \int_{t_0'}^t\right)\|e^{-(t-s)A}A^\alpha\|\Big[\|B_n u_n(s) - B_m u_m(s)\| \\
&\quad + \|C_n u_n(s-\tau) - C_m u_m(s-\tau)\| \\
&\quad + \int_{-\tau}^0 |a(\theta)|\|L_n u_n(s+\theta) - L_m u_m(s+\theta)\|\,\mathrm{d}\theta\Big]\mathrm{d}s.
\end{aligned} \tag{3.20}$$

Now, for $0 < \alpha < \beta < 1$, we have

$$\begin{aligned} &\|[B_n(u_n(s)) - B_m(u_m(s))]\| \\ &\quad \leq \|B_n(u_n(s)) - B_n(u_m(s))\| + \|B_n(u_m(s)) - B_m(u_m(s))\| \\ &\quad\quad \leq K_1(\eta_0)\|A^\alpha[u_n(s) - u_m(s)]\| + K_1(\eta_0)\|A^{\alpha-\beta}(P^n - P^m)A^\beta u_m(s)\| \\ &\quad\quad\quad \leq K_1(\eta_0)\|A^\alpha[u_n(s) - u_m(s)]\| + \frac{K_1(\eta_0)}{\lambda_m^{\beta-\alpha}}\|A^\beta u_m(s)\|. \end{aligned} \tag{3.21}$$

Similarly

$$\begin{aligned} &\|[C_n(u_n(s-\tau)) - C_m(u_m(s-\tau))]\| \\ &\quad \leq \|C_n(u_n(s-\tau)) - C_n(u_m(s-\tau))\| + \|C_n(u_m(s-\tau)) - C_m(u_m(s))\| \\ &\quad\quad \leq K_2(\eta_0)\|A^\alpha[u_n(s-\tau) - u_m(s-\tau)]\| + K_2(\eta_0)\|A^{\alpha-\beta}(P^n - P^m)A^\beta u_m(s-\tau)\| \\ &\quad\quad \leq K_2(\eta_0)\|A^\alpha[u_n(s-\tau) - u_m(s-\tau)]\| + \frac{K_2(\eta_0)}{\lambda_m^{\beta-\alpha}}\|A^\beta u_m(s-\tau)\| \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} &\|[L_n(u_n(s+\theta)) - L_m(u_m(s+\theta))]\| \\ &\quad \leq \|L_n(u_n(s+\theta)) - L_n(u_m(s+\theta))\| + \|L_n(u_m(s+\theta)) - L_m(u_m(s+\theta))\| \\ &\quad\quad \leq K_3(\eta_0)\|A^\alpha[u_n(s+\theta) - u_m(s+\theta)]\| + K_3(\eta_0)\|A^{\alpha-\beta}(P^n - P^m)A^\beta u_m(s+\theta)\| \\ &\quad\quad \leq K_3(\eta_0)\|A^\alpha[u_n(s+\theta) - u_m(s+\theta)]\| + \frac{K_3(\eta_0)}{\lambda_m^{\beta-\alpha}}\|A^\beta u_m(s+\theta)\|. \end{aligned} \tag{3.23}$$

We estimate the first term as

$$\begin{aligned} \|e^{-tA}A^\alpha(g_n(0,\tilde{\phi}) - g_m(0,\tilde{\phi}))\| &\leq M\|A^{\alpha-\beta}\|\,\|A^\beta g(0,P^n\phi) - A^\beta g(0,P^m\phi)\| \\ &\leq M\|A^{\alpha-\beta}\|L_g\|(P^n - P^m)A^\alpha\phi\|. \end{aligned}$$

The first and the third integrals are estimated as

$$\begin{aligned} &\int_0^{t_0'} \|A^{1+\alpha-\beta}e^{-(t-s)A}\|\,\|A^\beta g_n(s,u_n) - A^\beta g_m(s,u_m)\|\,\mathrm{d}s \\ &\qquad\qquad \leq 2C_{1+\alpha-\beta}(L_g\tilde{R} + B)(t_0 - t_0')^{-(1+\alpha-\beta)}t_0'. \end{aligned}$$

For the second and the fourth integrals, we have

$$\begin{aligned} &\int_{t_0'}^t \|A^{1+\alpha-\beta}e^{-(t-s)A}\|\,\|A^\beta g_n(s,u_n) - A^\beta g_m(s,u_m)\|\,\mathrm{d}s \\ &\quad \leq C_{1+\alpha-\beta}L_g\int_{t_0'}^t (t-s)^{-(1+\alpha-\beta)}\left[\|u_n(s) - u_m(s)\|_\alpha + \frac{1}{\lambda_m^{\beta-\alpha}}\|A^\vartheta u_m(s)\|\right]\mathrm{d}s \\ &\quad\quad \leq C_{1+\alpha-\beta}L_g\bigg(\frac{U_{t_0'}T^{\beta-\alpha}}{\lambda_m^{\beta-\alpha}(\beta-\alpha)} + \int_{t_0'}^t (t-s)^{-(1+\alpha-\beta)}\|u_n(s) - u_m(s)\|_\alpha\,\mathrm{d}s\bigg). \end{aligned}$$

From inequalities (3.21), (3.22) and (3.23), inequality (3.20) becomes

$$
\begin{aligned}
&\|u_n(t)-u_m(t)\|_\alpha \\
&\quad \le M\|A^{\alpha-\beta}\|L_g\|(P^n-P^m)A^\alpha\phi\| \\
&\quad + \|A^{\alpha-\beta}\|L_g\left(\|u_n(t)-u_m(t)\|_\alpha + \frac{U_{t_0'}}{\lambda_m^{\beta-\alpha}}\right) \\
&\quad + 2\left(\frac{C_{1+\alpha-\beta}(L_g\tilde{R}+B)}{(t_0-t_0')^{1+\alpha-\beta}}\right)t_0' + C_{\alpha,\beta}\frac{U_{t_0'}}{\lambda_m^{\vartheta-\alpha}} \\
&\quad + \int_{t_0'}^{t}\left(\frac{C_{1+\alpha-\beta}L}{(t-s)^{1+\alpha-\beta}}\right)\|u_n(s)-u_m(s)\|_\alpha\,\mathrm{d}s \\
&\quad + \left(\int_0^{t_0'}+\int_{t_0'}^{t}\right)\|e^{-(t-s)A}A^\alpha\|\Big[K_1(\eta_0)\|A^\alpha[u_n(s)-u_m(s)]\| \\
&\qquad + \frac{K_1(\eta_0)}{\lambda_m^{\beta-\alpha}}\|A^\beta u_m(s)\| + K_2(\eta_0)\|A^\alpha[u_n(s-\tau)-u_m(s-\tau)]\| \\
&\qquad + \frac{K_2(\eta_0)}{\lambda_m^{\beta-\alpha}}\|A^\beta u_m(s-\tau)\| + \int_{-\tau}^{0}\Big(|a(\theta)|K_3(\eta_0)\|A^\alpha[u_n(s+\theta)-u_m(s+\theta)]\| \\
&\qquad + \frac{K_3(\eta_0)}{\lambda_m^{\beta-\alpha}}\|A^\beta u_m(s+\theta)\|\Big)\mathrm{d}\theta\Big]\mathrm{d}s,
\end{aligned}
\tag{3.24}
$$

where $C_{\alpha,\beta} = C_{1+\alpha-\beta}L_g\frac{T^{\beta-\alpha}}{\beta-\alpha}$. Since $\|A^{\alpha-\beta}\|L_g < 1$, and from Corollaries 1 and 2, inequality (3.24) implies that

$$
\begin{aligned}
&\|u_n(t)-u_m(t)\|_\alpha \\
&\quad \le \frac{1}{(1-\|A^{\alpha-\beta}\|L_g)}\bigg\{M\|(P^n-P^m)A^\alpha\phi\| + \|A^{\alpha-\beta}\|L_g\frac{U_{t_0'}}{\lambda_m^{\beta-\alpha}} \\
&\qquad + 2\left(\frac{C_{1+\alpha-\beta}(L_g\tilde{R}+B)}{(t_0-t_0')^{1+\alpha-\beta}}\right)t_0' + C_{\alpha,\beta}\frac{U_{t_0'}}{\lambda_m^{\beta-\alpha}} + C_1.t_0' + \frac{C_2}{\lambda_m^{\beta-\alpha}} \\
&\qquad + \big(C_\alpha K(\eta_0)+C_{1+\alpha-\beta}\big)\int_{t_0'}^{t}\left((t-s)^{-\alpha} + \frac{L_g}{(t-s)^{1+\alpha-\beta}}\right)\|u_n-u_m\|_{s,\alpha}\,\mathrm{d}s\bigg\},
\end{aligned}
\tag{3.25}
$$

where $C_1 = 2C_\alpha(t_0-t_0')^{-\alpha}CK(\eta_0)$ and $C_2 = \frac{2K(\eta_0)C_\alpha T^{1-\alpha}}{(1-\alpha)}$. Now, replacing t by $t+\theta$ in inequality (3.25), where $\theta \in [t_0'-t, 0]$, we get

$$
\begin{aligned}
&\|u_n(t+\theta)-u_m(t+\theta)\|_\alpha \\
&\le \frac{1}{(1-\|A^{\alpha-\beta}\|L)}\bigg\{M\|(P^n-P^m)A^\alpha\phi\| + \|A^{\alpha-\beta}\|L\frac{U_{t_0'}}{\lambda_m^{\beta-\alpha}} \\
&\quad + 2\left(\frac{C_{1+\alpha-\beta}(L_g\tilde{R}+B)}{(t_0-t_0')^{1+\alpha-\beta}}\right)t_0' + C_{\alpha,\beta}\frac{U_{t_0'}}{\lambda_m^{\beta-\alpha}} + C_1.t_0' + \frac{C_2}{\lambda_m^{\beta-\alpha}} \\
&\quad + \big(C_\alpha K(\eta_0)+C_{1+\alpha-\beta}\big)\int_{t_0'}^{t+\theta}\left((t+\theta-s)^{-\alpha} + \frac{L_g}{(t+\theta-s)^{1+\alpha-\beta}}\right)\|u_n-u_m\|_{s,\alpha}\,\mathrm{d}s\bigg\}.
\end{aligned}
\tag{3.26}
$$

We put $s-\theta=\gamma$ in (3.26) to get

$$\begin{aligned}
&\|u_n(t+\theta)-u_m(t+\theta)\|_\alpha \\
&\quad\le \frac{1}{(1-\|A^{\alpha-\beta}\|L_g)}\bigg\{M\|(P^n-P^m)A^\alpha\phi\|+\|A^{\alpha-\beta}\|L_g\frac{U_{t_0'}}{\lambda_m^{\beta-\alpha}} \\
&\qquad+2\bigg(\frac{C_{1+\alpha-\beta}(L_g\tilde{R}+B)}{(t_0-t_0')^{1+\alpha-\beta}}\bigg)t_0' \\
&\qquad+C_{\alpha,\beta}\frac{U_{t_0'}}{\lambda_m^{\beta-\alpha}}+C_1.t_0'+\frac{C_2}{\lambda_m^{\beta-\alpha}} \\
&\qquad+\big(C_\alpha K(\eta_0)+C_{1+\alpha-\beta}\big)\int_{t_0'-\theta}^t\bigg((t-\gamma)^{-\alpha}+\frac{L_g}{(t-\gamma)^{1+\alpha-\beta}}\bigg)\|u_n-u_m\|_{\gamma,\alpha}\,\mathrm{d}\gamma\bigg\} \\
&\quad\le \frac{1}{(1-\|A^{\alpha-\beta}\|L)}\bigg\{M\|(P^n-P^m)A^\alpha\phi\|+\|A^{\alpha-\beta}\|L_g\frac{U_{t_0'}}{\lambda_m^{\beta-\alpha}} \\
&\qquad+2\bigg(\frac{C_{1+\alpha-\beta}(L_g\tilde{R}+B)}{(t_0-t_0')^{1+\alpha-\beta}}\bigg)t_0' \\
&\qquad+C_{\alpha,\beta}\frac{U_{t_0'}}{\lambda_m^{\beta-\alpha}}+C_1.t_0'+\frac{C_2}{\lambda_m^{\beta-\alpha}} \\
&\qquad+\big(C_\alpha K(\eta_0)+C_{1+\alpha-\beta}\big)\int_{t_0'}^t\bigg((t-\gamma)^{-\alpha}+\frac{L_g}{(t-\gamma)^{1+\alpha-\beta}}\bigg)\|u_n-u_m\|_{\gamma,\alpha}\,\mathrm{d}\gamma\bigg\}.
\end{aligned}$$

Now

$$\begin{aligned}
&\sup_{t_0'-t\le\theta\le0}\|u_n(t+\theta)-u_m(t+\theta)\|_\alpha \\
&\quad\le \frac{1}{(1-\|A^{\alpha-\beta}\|L_g)}\bigg\{M\|(P^n-P^m)A^\alpha\phi\|+\|A^{\alpha-\beta}\|L\frac{U_{t_0'}}{\lambda_m^{\beta-\alpha}} \\
&\qquad+2\bigg(\frac{C_{1+\alpha-\beta}(L_g\tilde{R}+B)}{(t_0-t_0')^{1+\alpha-\beta}}\bigg)t_0' \\
&\qquad+C_{\alpha,\beta}\frac{U_{t_0'}}{\lambda_m^{\beta-\alpha}}+C_1.t_0'+\frac{C_2}{\lambda_m^{\beta-\alpha}} \\
&\qquad+\big(C_\alpha K(\eta_0)+C_{1+\alpha-\beta}\big)\int_{t_0'}^t\bigg((t-\gamma)^{-\alpha}+\frac{L_g}{(t-\gamma)^{1+\alpha-\beta}}\bigg)\|u_n-u_m\|_{\gamma,\alpha}\,\mathrm{d}\gamma\bigg\}.
\end{aligned}\tag{3.27}$$

We have

$$\begin{aligned}
&\sup_{-\tau-t\le\theta\le0}\|u_n(t+\theta)-u_m(t+\theta)\|_\alpha \\
&\qquad\le \sup_{0\le\theta+t\le t_0'}\|u_n(t+\theta)-u_m(t+\theta)\|_\alpha+\sup_{t_0'-t\le\theta\le0}\|u_n(t+\theta)-u_m(t+\theta)\|_\alpha.
\end{aligned}$$

Using inequalities (3.27) and (3.24) in the above inequality, we get

$$\begin{aligned}
&\sup_{-\tau\le t+\theta\le t}\|u_n(t+\theta)-u_m(t+\theta)\|_\alpha\\
&\quad\le \frac{1}{(1-\|A^{\alpha-\beta}\|L_g)}\bigg\{M\|(P^n-P^m)A^\alpha\phi\|+\|A^{\alpha-\beta}\|L_g\frac{U_{t_0'}}{\lambda_m^{\beta-\alpha}}\\
&\quad+2\bigg(\frac{C_{1+\alpha-\beta}(L_g\tilde{R}+B)}{(t_0-t_0')^{1+\alpha-\beta}}\bigg)t_0'\\
&\quad+C_{\alpha,\beta}\frac{U_{t_0'}}{\lambda_m^{\beta-\alpha}}+(C_1+C_3).t_0'+\frac{C_2+C_4}{\lambda_m^{\beta-\alpha}}\\
&\quad+\big(C_\alpha K(\eta_0)+C_{1+\alpha-\beta}\big)\int_{t_0'}^{t}\bigg((t-\gamma)^{-\alpha}+\frac{L_g}{(t-\gamma)^{1+\alpha-\beta}}\bigg)\|u_n-u_m\|_{\gamma,\alpha}\,\mathrm{d}\gamma\bigg\},
\end{aligned}$$

where C_3 and C_4 are constants. An application of Gronwall's inequality to the above inequality gives the required result. This completes the proof of Theorem 2. □

With the help of Theorems 1 and 2, we may state the following existence, uniqueness and convergence result.

Theorem 3 *Suppose that the conditions* **(H1)–(Hg)** *are satisfied and* $h(t)\in D(A)$ *for all* $t\in[-\tau,0]$. *Then there exist a function* $u_n\in C([-\tau,T_0];H)$ *and* $u\in C([-\tau,T_0];H)$ *satisfying*

$$u_n(t)=\begin{cases} h(t), & t\in[-\tau,0],\\ e^{-tA}\big(h(0)+g_n(0,h(0))\big)-g_n(t,u_n(t)) & \\ \quad+\displaystyle\int_0^t Ae^{-(t-s)A}g_n(s,u_n)\,\mathrm{d}s+\int_0^t e^{-(t-s)A}\Big[B_nu_n(s) & \\ \qquad+C_nu_n(s-\tau)+\displaystyle\int_{-\tau}^0 a(\theta)L_nu_n(s+\theta)\,\mathrm{d}\theta\Big]\mathrm{d}s, & t\in[0,T_0]\end{cases}\tag{3.28}$$

and

$$u(t)=\begin{cases} h(t), & t\in[-\tau,0],\\ e^{-tA}\big(h(0)+g(0,h(0))\big)-g(t,u(t)) & \\ \quad+\displaystyle\int_0^t Ae^{-(t-s)A}g(s,u(s))\,\mathrm{d}s+\int_0^t e^{-(t-s)A}\Big[Bu(s) & \\ \qquad+Cu(s-\tau)+\displaystyle\int_{-\tau}^0 a(\theta)L_gu(s+\theta)\,\mathrm{d}\theta\Big]\mathrm{d}s, & t\in[0,\tilde{T}]\end{cases}\tag{3.29}$$

such that $u_n\to u$ *in* $C([-\tau,T_0];H)$ *as* $n\to\infty$, *where* B_n, C_n *and* L_n *are as defined earlier.*

4 Faedo–Galerkin approximations

We know from the previous sections that for any $-\tau \leq T_0 \leq T$ we have a unique $u \in C^{\alpha}_{T_0}$ satisfying the integral equation

$$u(t) = \begin{cases} h(t), & t \in [-\tau, 0], \\ e^{-tA}\big(h(0) + g(0, h(0))\big) - g(t, u(t)) & \\ \quad + \int_0^t Ae^{-(t-s)A} g(s, u(s))\, \mathrm{d}s + \int_0^t e^{-(t-s)A}\Big[Bu(s) & \\ \quad\quad + Cu(s-\tau) + \int_{-\tau}^0 a(\theta) L_g u(s+\theta)\, \mathrm{d}\theta\Big] \mathrm{d}s, & t \in [0, \tilde{T}]. \end{cases} \tag{4.1}$$

Also, there is a unique solution $u \in C^{\alpha}_{T_0}$ of the approximate integral equation

$$u_n(t) = \begin{cases} h(t), & t \in [-\tau, 0], \\ e^{-tA}\big(h(0) + g_n(0, h(0))\big) - g_n(t, u_n(t)) & \\ \quad + \int_0^t Ae^{-(t-s)A} g_n(s, u_n)\, \mathrm{d}s + \int_0^t e^{-(t-s)A}\Big[B_n u_n(s) & \\ \quad\quad + C_n u_n(s-\tau) + \int_{-\tau}^0 a(\theta) L_n u_n(s+\theta) d\theta\Big] \mathrm{d}s, & t \in [0, T_0]. \end{cases} \tag{4.2}$$

Faedo–Galerkin approximation $\bar{u}_n = P^n u_n$ is given by

$$\bar{u}_n(t) = \begin{cases} P^n h(t), & t \in [-\tau, 0], \\ e^{-tA} P^n\big(h(0) + g_n(0, h(0))\big) - P_n g_n(t, u_n(t)) & \\ \quad + \int_0^t Ae^{-(t-s)A} P_n g_n(s, u_n)\, \mathrm{d}s + \int_0^t e^{-(t-s)A} P^n\Big[B_n u_n(s) & \\ \quad\quad + C_n u_n(s-\tau) + \int_{-\tau}^0 a(\theta) L_n u_n(s+\theta)\, \mathrm{d}\theta\Big] \mathrm{d}s, & t \in [0, T_0], \end{cases} \tag{4.3}$$

where B_n, C_n and L_n are as defined earlier.

If the solution $u(t)$ to (4.1) exists on $-\tau \leq t \leq T_0$, then it has the representation

$$u(t) = \sum_{i=0}^{\infty} \alpha_i(t) \phi_i, \tag{4.4}$$

where $\alpha_i(t) = (u(t), \phi_i)$ for $i = 0, 1, 2, 3, \cdots$ and

$$\bar{u}_n(t) = \sum_{i=0}^{n} \alpha_i^n(t) \phi_i, \tag{4.5}$$

where $\alpha_i^n(t) = (\bar{u}_n(t), \phi_i)$ for $i = 0, 1, 2, 3, \cdots$.

As a consequence of Theorem 1 and Theorem 2, we have the following result.

Theorem 4 *Suppose that the conditions* **(H1)–(Hg)** *are satisfied and* $h(t) \in D(A)$ *for all* $t \in [-\tau, 0]$. *Then there exist unique functions* $\bar{u}_n \in C([-\tau, T_0]; H_n)$ *and* $u \in C([-\tau, T_0]; H)$ *satisfying*

$$\bar{u}_n(t) = \begin{cases} P^n h(t), & t \in [-\tau, 0], \\ e^{-tA} P^n \big(h(0) + g_n(0, h(0))\big) - P_n g_n(t, u_n(t)) & \\ \quad + \int_0^t A e^{-(t-s)A} P_n g_n(s, u_n)\, \mathrm{d}s + \int_0^t e^{-(t-s)A} P^n \Big[B_n u_n(s) & \\ \quad + C_n u_n(s-\tau) + \int_{-\tau}^0 a(\theta) L_n u_n(s+\theta)\, \mathrm{d}\theta \Big] \mathrm{d}s, & t \in [0, T_0] \end{cases}$$

and

$$u(t) = \begin{cases} h(t), & t \in [-\tau, 0], \\ e^{-tA} \big(h(0) + g(0, h(0))\big) - g(t, u(t)) & \\ \quad + \int_0^t A e^{-(t-s)A} g(s, u(s))\, \mathrm{d}s + \int_0^t e^{-(t-s)A} \Big[B u(s) & \\ \quad + C u(s-\tau) + \int_{-\tau}^0 a(\theta) L_g u(s+\theta) d\theta \Big] \mathrm{d}s, & t \in [0, \tilde{T}] \end{cases}$$

such that $\bar{u}_n \to u$ *in* $C([-\tau, T_0]; H)$ *as* $n \to \infty$, *where* B_n, C_n *and* L_n *are as defined earlier.*

Theorem 5 *Let* **(H1)–(H6)** *hold. If* $h(t) \in D(A)$ *for all* $t \in [-\tau, 0]$, *then for any* $-\tau \le t \le T_0 \le T$,

$$\lim_{n\to\infty} \sup_{-\tau \le t \le T_0} \left[\sum_{i=0}^{n} \lambda_i^{2\alpha} \{\alpha_i(t) - \alpha_i^n(t)\}^2 \right] = 0.$$

Proof. Let $\alpha_i^n(t) = 0$ for $i > n$. We have

$$A^\alpha [u(t) - \bar{u}_n(t)] = A^\alpha \left[\sum_{i=0}^{\infty} \{\alpha_i(t) - \alpha_i^n(t)\} \phi_i \right] = \sum_{i=0}^{\infty} \lambda_i^\alpha \{\alpha_i(t) - \alpha_i^n(t)\} \phi_i.$$

Thus, we have

$$\|A^\alpha [u(t) - \bar{u}_n(t)\|^2 \ge \sum_{i=0}^{n} \lambda_i^{2\alpha} |\alpha_i(t) - \alpha_i^n(t)|^2. \tag{4.6}$$

Hence, as a consequence of Theorem 3, we have the required result. □

5 Example

Let us consider a quantum particle in a potential $V(x)$. Then the particle is represented by its wave function $u(t,x)$ which is a solution of the following partial differential equation with delay of Schrödinger type

$$\begin{aligned} i\frac{\partial(u(t,x)+g(t,x))}{\partial t} &= (-\triangle + V(x))u(t,x) + b(u(t,x))u_x(t,x) \\ &\quad + c(u(t-\tau,x))u_x(t-\tau,x) \\ &\quad + \int_{-\tau}^{0} a(s)l(u(t+s,x))u_x(t+s,x)\,\mathrm{d}s, \quad t \geq 0,\ x \in (0,1), \end{aligned} \tag{5.1}$$

$$\begin{aligned} u(t,x) &= \tilde{h}(t,x), \quad t \in [-\tau,0],\ x \in (0,1), \\ u(t,0) &= u(t,1) = 0, \quad t \geq 0, \end{aligned}$$

where the kernel $a \in L^p_{\text{loc}}(-\tau,0)$, b, c, l are smooth functions from $\mathbb{R}$ into $\mathbb{R}$, $\tilde{h}$ is a given continuous function and $\tau > 0$ is a given number.

We define the operator A as follows:

$$Au = (i\triangle - iV)u \quad \text{with} \quad u \in D(A) = H^1_0(0,1) \cap H^2(0,1). \tag{5.2}$$

Clearly, the operator A satisfies the hypothesis **(H1)** and is the infinitesimal generator of an analytic semigroup $\{e^{-tA} : t \geq 0\}$.

For $0 \leq \alpha < 1$ and $t \in [0,T]$, we denote $C^\alpha_t := C([-\tau,t]; D(A^\alpha))$, which is the Banach space endowed with the sup norm

$$\|\psi\|_{t,\alpha} := \sup_{-\tau \leq \eta \leq t} \|\psi(\eta)\|_\alpha.$$

We observe some properties of the operators A and A^α defined by (5.2). For $\phi \in D(A)$ and $\lambda \in \mathbb{R}$, with $A\phi = (i\triangle - iV)u = \lambda u$, we have $\langle A\phi, \phi\rangle = \langle \lambda\phi, \phi\rangle$, that is,

$$\left\langle -\phi'', \phi \right\rangle = |u'|^2_{L^2} = \lambda|\phi|^2_{L^2},$$

so $\lambda > 0$. For $u \in D(A)$ there exists a sequence of real numbers $\{\alpha_n\}$ such that

$$u(x) = \sum_{n\in\mathbb{N}} \alpha_n \phi_n(x), \quad \sum_{n\in\mathbb{N}} (\alpha_n)^2 < +\infty \quad \text{and} \quad \sum_{n\in\mathbb{N}} (\lambda_n)^2(\alpha_n)^2 < +\infty.$$

We have

$$A^{1/2}u(x) = \sum_{n\in\mathbb{N}} \sqrt{\lambda_n}\, \alpha_n\, \phi_n(x)$$

with $u \in D(A^{1/2}) = H^1_0(0,1)$, that is, $\sum_{n\in\mathbb{N}} \lambda_n(\alpha_n)^2 < +\infty$.

Equation (5.1) can be reformulated as the following abstract equation in a separable Hilbert space $H = L^2(0,1)$:

$$\begin{aligned} \frac{\partial(w(t)+g(t,w(t)))}{\partial t} + Aw(t) &= Bw(t) + Cw(t-\tau) + \int_{-\tau}^{0} a(\theta)Lw(t+\theta)\,\mathrm{d}\theta, \\ &\qquad 0 < t \leq T < \infty,\ \tau > 0, \\ w(t) &= h(t), \quad t \in [-\tau,0], \end{aligned}$$

where $w(t) = u(t, \cdot)$, that is, $w(t)(x) = u(t, x)$, $w_t(\theta)(x) = u(t + \theta, x)$, $t \in [0, T]$, $\theta \in [-\tau, 0]$, $x \in (0, 1)$, the operator A is defined as in equation (5.2) and $h(\theta)(x) = \tilde{h}(\theta, x)$ for all $\theta \in [-\tau, 0]$ and $x \in (0, 1)$. The operators B, C and L are defined as follows:

- $B\colon D(A^{1/2}) \to H$, where $Bw(t)(x) = -ib(u(t, x))u_x(t, x)$;
- $C\colon D(A^{1/2}) \to H$, where $Cw(t - \tau)(x) = -ic(u(t - \tau, x))u_x(t - \tau, x)$;
- $L\colon D(A^{1/2}) \to H$, where $Lw(t + s)(x) = -il(u(t + s, x))u_x(t + s, x)$, where $s \in [-\tau, 0]$ and $x \in (0, 1)$.

Let α be such that $3/4 < \alpha < 1$. For $u, v \in D(A^\alpha)$ with $\|A^\alpha u\| \leq \eta$ and $\|A^\alpha v\| \leq \eta$, we have

$$\begin{aligned}
&|b(u(x))u_x(x) - b(v(x))v_x(x)| \\
&\quad \leq |b(u(x)) - b(v(x))||u_x(x)| + |b(v(x))||u_x(x) - v_x(x)| \\
&\quad \leq L_b|u(x) - v(x)||u_x(x)| + b_1|u_x(x) - v_x(x)|,
\end{aligned}$$

where L_b is the Lipschitz constant for b and $b_1 = L_b \frac{\eta}{\lambda_0^{1/2}} + |b(0)|$. For $u, v \in D(A^\alpha) \subset D(A^{1/2})$, we have

$$\|B(u) - B(v)\|^2 \leq \int_0^1 \big| \big[L_b|u(x) - v(x)||u_x(x)| + b_1|u_x(x) - v_x(x)|\big] \big|^2 \, \mathrm{d}x.$$

Thus, from [16, Lemma 8.3.3], we get

$$\begin{aligned}
\|B(u) - B(v)\|^2 &\leq 2{L_b}^2 \int_0^1 |u(x) - v(x)|^2 |u_x(x)|^2 \, \mathrm{d}x + 2{b_1}^2 \int_0^1 |u_x(x) - v_x(x)||^2 \, \mathrm{d}x \\
&\leq 2{L_b}^2 \|u - v\|_\infty^2 \int_0^1 |u_x(x)|^2 \, \mathrm{d}x + 2{b_1}^2 \int_0^1 |u_x(x) - v_x(x)|^2 \, \mathrm{d}x \\
&\leq 2{L_b}^2 \|u - v\|_\infty^2 \|A^{1/2}u\|^2 + 2{b_1}^2 \|A^{1/2}(u - v)\|^2 \\
&\leq 2{L_b}^2 c^2 \eta^2 \|A^\alpha(u - v)\|^2 + 2{b_1}^2 \|A^\alpha(u - v)\|^2 \\
&\leq M_b(\eta)^2 \|A^\alpha(u - v)\|^2,
\end{aligned}$$

where $3/4 < \alpha < 1$, $\|A^\alpha u\| \leq \eta$, $\|A^\alpha v\| \leq \eta$, $M_b(\eta) = \sqrt{2}[L_b c\eta + b_1]$, $\|u\|_\infty = \sup_{0 \leq x \leq 1} |u(x)|$ and $\|u\|_\infty \leq c\|A^\alpha u\|$ for any $u \in D(A^\alpha)$. Hence, the operator B restricted to $D(A^\alpha)$ satisfies the hypothesis **(H3)** with $K_1(\eta) = M_b(\eta)$. Similarly, we can show that the operators C and L satisfy the hypotheses **(H4)** and **(H5)**, respectively.

The nonlinear operators of the above type arise in the theory of shock waves, or various types of Schrödinger equations, turbulence and continuous stochastic processes (cf. [9] for more details).

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