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EXISTENCE OF μ-PSEUDO ALMOST AUTOMORPHIC SOLUTIONS TO A NONAUTONOMOUS SEMILINEAR EVOLUTION EQUATION BY INTERPOLATION THEORY

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Abstract. In this paper, we shall deal with the existence and uniqueness of μ-pseudo almost automorphic solutions to a nonautonomous semilinear evolution equation in Banach spaces. We obtain our main results by the interpolation theory and properties of μ-pseudo almost automorphic functions.

Keywords: μ-pseudo almost automorphic function, nonautonomous semilinear evolution equation, interpolation theory.

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1 Introduction

The concept of almost automorphy was first introduced in the literature by Bochner in [8], which is a natural generalization of almost periodicity [9]. For more details about this topic, we refer to [14, 21, 22]. The concept of asymptotically almost automorphic functions was introduced by N'Guérékata in [20]. Liang, Xiao and Zhang in [16, 25] presented the concept of pseudo almost automorphy. In [23], N'Guérékata and Pankov introduced another generalization of almost automorphic functions-Stepanov-like almost automorphic functions. Such a notation, subsequently, was applied to investigate the existence of weak Stepanov-like almost automorphic solutions to

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some parabolic evolution equations. Blot et al. introduced the notion of weighted pseudo almost automorphic functions in [6], which generalizes that of pseudo-almost automorphic functions. Xia and Fan presented the notation of Stepanov-like weighted pseudo almost automorphic function in [24]. Zhang, Chang and N'Guérékata investigated some properties and new composition theorems of Stepanov-like weighted pseudo almost automorphic functions in [27, 28].

Recently, Blot, Cieutat and Ezzinbi in [7] applied the measure theory to define an ergodic function and they investigated many interesting properties of μ-pseudo almost automorphic functions. Thus, the classical theory of pseudo almost automorphy becomes a particular case of their approach in [7]. In this work, we investigate the existence of μ-pseudo almost automorphic mild solutions to the following nonautonomous semilinear evolution equation:

$$u'(t) = A(t)u(t) + f(t, u(t-h)), \qquad t \in \mathbb{R}, \tag{1.1}$$

where $h \geq 0$ is a fixed constant, and $\{A(t)\}_{t\in\mathbb{R}}$ satisfies the Acquistapace-Terreni condition in [1], $U(t,s)$ generated by $A(t)$ is exponentially stable, and $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is a suitable continuous function. Our main results are based upon the interpolation theory developed in [11, 12, 13].

The rest of this paper is organized as follows. In section 2, we present some basic definitions, lemmas, and preliminary results which will be used throughout this paper. In section 3, we prove the existence and uniqueness of μ-pseudo almost automorphic mild solutions to the nonautonomous semilinear evolution equation (1.1).

2 Preliminaries

This section is devoted to some preliminary results needed in the sequel. Throughout the paper, the notations $(\mathbb{X}, \|\cdot\|)$ and $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ are two Banach spaces and $BC(\mathbb{R}, \mathbb{X})$ denotes the Banach space of all bounded continuous functions from $\mathbb{R}$ to $\mathbb{X}$, equipped with the supremum norm $\|f\|_\infty = \sup_{t\in\mathbb{R}} \|f(t)\|$. Let $\mathbb{X}_\alpha$ be an intermediate space between $D(A)$ and $\mathbb{X}$. $B(\mathbb{R}, \mathbb{X}_\alpha)$ for $\alpha \in (0,1)$ stands for the Banach space of all bounded continuous functions $\varphi : \mathbb{R} \to \mathbb{X}_\alpha$ when equipped with the α-sup norm:

$$\|\varphi\|_{\alpha,\infty} := \sup_{t\in\mathbb{R}} \|\varphi(t)\|_\alpha$$

for $\varphi \in BC(\mathbb{R}, \mathbb{X}_\alpha)$.

Throughout this work, we denote by $\mathcal{B}$ the Lebesgue σ-field of $\mathbb{R}$ and by $\mathcal{M}$ the set of all positive measures μ on $\mathcal{B}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a,b]) < +\infty$, for all $a, b \in \mathbb{R}(a < b)$.

Definition 2.1 *[8] A continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be almost automorphic if for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that*

$$g(t) := \lim_{n\to\infty} f(t + s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t - s_n) = f(t)$$

for each $t \in \mathbb{R}$. The collection of all such functions will be denoted by $AA(\mathbb{X})$.

Definition 2.2 *[26] A continuous function $f(t,s) : \mathbb{R} \times \mathbb{R} \to \mathbb{X}$ is called bi-almost automorphic if for every sequence of real numbers $\{s_n'\}_{n\in\mathbb{N}}$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that*

$$g(t,s) := \lim_{n\to\infty} f(t+s_n, s+s_n)$$

is well defined for each $t, s \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t-s_n, s-s_n) = f(t,s)$$

for each $t, s \in \mathbb{R}$. The collection of all such functions will be denoted by $bAA(\mathbb{R}\times\mathbb{R},\mathbb{X})$.

Define

$$PAA_0(\mathbb{R},\mathbb{X}) = \left\{ \phi \in BC(\mathbb{R},\mathbb{X}) : \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|\phi(\sigma)\| \, \mathrm{d}\sigma = 0 \right\}.$$

In the same way, we define $PAA_0(\mathbb{R}\times\mathbb{X},\mathbb{X})$ as the collection of jointly continuous functions $f : \mathbb{R}\times\mathbb{X} \to \mathbb{X}$ which belong to $BC(\mathbb{R}\times\mathbb{X},\mathbb{X}))$ and satisfy

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|\phi(\sigma,x)\| \, \mathrm{d}\sigma = 0$$

uniformly in compact subset of $\mathbb{X}$.

Definition 2.3 *[17, 26] A continuous function $f : \mathbb{R} \to \mathbb{X}$ (respectively $\mathbb{R}\times\mathbb{X} \to \mathbb{X}$) is called pseudo-almost automorphic if it can be decomposed as $f = g + \phi$, where $g \in AA(\mathbb{R},\mathbb{X})$(respectively $AA(\mathbb{R}\times\mathbb{X},\mathbb{X})$) and $\phi \in PAA_0(\mathbb{R},\mathbb{X})$(respectively $PAA_0(\mathbb{R}\times\mathbb{X},\mathbb{X})$). Denote by $PAA(\mathbb{R},\mathbb{X})$ (respectively $PAA(\mathbb{R}\times\mathbb{X},\mathbb{X})$) the set of all such functions.*

Definition 2.4 *[7] Let $\mu \in \mathcal{M}$. A bounded continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be μ-ergodic if*

$$\lim_{r\to+\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \|f(t)\| \, \mathrm{d}\mu(t) = 0.$$

We denote the space of all such functions by $\varepsilon(\mathbb{R},\mathbb{X},\mu)$.

Definition 2.5 *[7] Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be μ-pseudo almost automorphic if f is written in the form: $f = g + \phi$, where $g \in AA(\mathbb{R},\mathbb{X})$ and $\phi \in \varepsilon(\mathbb{R},\mathbb{X},\mu)$. We denote the space of all such functions by $PAA(\mathbb{R},\mathbb{X},\mu)$.*

Obviously, we have $AA(\mathbb{R},\mathbb{X}) \subset PAA(\mathbb{R},\mathbb{X},\mu) \subset BC(\mathbb{R},\mathbb{X})$.

Lemma 2.1 *[7, Proposition 2.13] Let $\mu \in \mathcal{M}$. Then $(\varepsilon(\mathbb{R},\mathbb{X},\mu), \|\cdot\|_\infty)$ is a Banach space.*

Lemma 2.2 *[7, Theorem 4.1] Let $\mu \in \mathcal{M}$ and $f \in PAA(\mathbb{R},\mathbb{X},\mu)$ be such that $f = g + \phi$, where $g \in AA(\mathbb{R},\mathbb{X})$ and $\phi \in \varepsilon(\mathbb{R},\mathbb{X},\mu)$. If $PAA(\mathbb{R},\mathbb{X},\mu)$ is translation invariant, then $\{g(t) : t \in \mathbb{R}\} \subset \overline{\{f(t) : t \in \mathbb{R}\}}$, (the closure of the range of f).*

Lemma 2.3 *[7, Theorem 2.14] Let $\mu \in \mathcal{M}$ and I be a bounded interval (eventually $I = \emptyset$). Assume that $f \in BC(\mathbb{R}, \mathbb{X})$. Then the following assertions are equivalent:*

(i) $f \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$;

(ii) $\lim_{r\to+\infty} \frac{1}{\mu([-r,r]\setminus I)} \int_{[-r,r]\setminus I} \|f(t)\| \,\mathrm{d}\mu(t) = 0$;

(iii) For any $\varepsilon > 0, \lim_{r\to+\infty} \frac{\mu(\{t\in[-r,r]\setminus I : \|f(t)\|>\varepsilon\})}{\mu([-r,r]\setminus I)} = 0.$

Lemma 2.4 *[7, Theorem 4.7] Let $\mu \in \mathcal{M}$. Assume that $PAA(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant. Then the decomposition of a μ-pseudo almost automorphic function in the form $f = g + \phi$ where $g \in AA(\mathbb{R}, \mathbb{X})$ and $\phi \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$, is unique.*

Lemma 2.5 *[7, Theorem 4.9] Let $\mu \in \mathcal{M}$. Assume that $PAA(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant. Then $(PAA(\mathbb{R}, \mathbb{X}, \mu), \|\cdot\|_\infty)$ is a Banach space.*

Theorem 2.1 *[10] Let $\mu \in \mathcal{M}$ and $f = g + h \in PAA(\mathbb{R} \times \mathbb{X}, \mathbb{X}, \mu)$. Assume that*

(a1) $f(t, x)$ is uniformly continuous on any bounded subset $K \subset \mathbb{X}$ uniformly in $t \in \mathbb{R}$.

(a2) $g(t, x)$ is uniformly continuous on any bounded subset $K \subset \mathbb{X}$ uniformly in $t \in \mathbb{R}$.

Then the function defined by $F(\cdot) := f(\cdot, \phi(\cdot)) \in PAA(\mathbb{R}, \mathbb{X}, \mu)$ if $\phi \in PAA(\mathbb{R}, \mathbb{X}, \mu)$.

Now, we introduce some notions and properties about evolution families and intermediate spaces.

Let $\mathbb{X}$ and $\mathbb{Z}$ be Banach spaces, with norms $\|\cdot\|$, $\|\cdot\|_{\mathbb{Z}}$ respectively, and suppose that $\mathbb{Z}$ is continuously embedded in $\mathbb{X}$, that is, $\mathbb{Z} \hookrightarrow \mathbb{X}$.

(H1) The family of closed linear operators $A(t)$ for $t \in \mathbb{R}$ on $\mathbb{X}$ with domain $D(A(t))$ (possibly not densely defined) satisfy the so-called Acquistapace-Terreni conditions, that is, there exist constants $\omega \in \mathbb{R}$, $\theta \in (\pi/2, \pi)$, $\mathscr{L}$, $\mathscr{K} > 0$ and $\mu, v \in (0, 1]$ with $\mu + v > 1$ such that

$$\Sigma_\theta \cup \{0\} \subset \rho(A(t) - \omega) \ni \lambda, \ \ \|R(\lambda, A(t) - \omega)\| \leq \frac{\mathscr{K}}{1 + |\lambda|} \quad \text{for all } \ t \in \mathbb{R}$$

and

$$\|(A(t) - \omega)R(\lambda, A(t) - \omega)[R(\omega, A(t)) - R(\omega, A(s))]\| \leq \mathscr{L}\frac{|t - s|^\mu}{|\lambda|^v}$$

for $t, s \in \mathbb{R}$, $\lambda \in \Sigma_\theta := \{\lambda \in \mathbb{C}\setminus\{0\} : |\arg \lambda| \leq \theta\}$.

Among other things, Acquistapace-Terreni Conditions ensure that there exists a unique evolution family:

$$\mathcal{U} = \{U(t, s) : t, s \in \mathbb{R} \ \text{ such that } \ t \geq s\}$$

on $\mathbb{X}$ associated with $A(t)$ such that $U(t, s)\mathbb{X} \subseteq D(A(t))$ for all $t, s \in \mathbb{R}$ with $t \geq s$, and
(a) $U(t, s)U(s, r) = U(t, r)$ for $t, s \in \mathbb{R}$ such that $t \geq s \geq r$;
(b) $U(t, t) = I$ for $t \in \mathbb{R}$ where I is the identity operator of $\mathbb{X}$;
(c) $(t, s) \to U(t, s) \in B(\mathbb{X})$ is continuous for $t > s$.

Definition 2.6 *(Baroun et al. [4]) An evolution family $\mathcal{U}$ is said to have an exponential dichotomy (or is hyperbolic) if there are projections $P(t)(t \in \mathbb{R})$ that are uniformly bounded and strongly*

continuous in t *and constants* $\delta > 0$ *and* $N \geq 1$ *such that*
(e) $U(t,s)\,P(s) = P(t)\,U(t,s)$*;*
(f) The restriction $U_Q(t,s) : Q(s)\mathbb{X} \to Q(t)\mathbb{X}$ *of* $U(t,s)$ *is invertible (we then set* $\widetilde{U}_Q(s,t) := U_Q(t,s)^{-1}$*); and*
(g) $\|U(t,s)\,P(s)\| \leq Ne^{-\delta(t-s)}$ *and* $\|\widetilde{U}_Q(s,t)\,Q(t)\| \leq Ne^{-\delta(t-s)}$ *for* $t \geq s$ *and* $t,\, s \in \mathbb{R}$*.*

In what follows, we introduce the interpolation spaces for $A(t)$. The following facts are most from the monograph [14]. One can also refer to [2, 15, 18] for further details.

Let A be a sectorial operator on $\mathbb{X}$ (Assumption (H1) holds when $A(t)$ is replaced with A) and let $\alpha \in (0,1)$. Define the real interpolation space:

$$\mathbb{X}_\alpha^A := \{x \in \mathbb{X} : \|x\|_\alpha^A := \sup_{r>0} \|r^\alpha (A-\omega) R(r, A-\omega)x\| < \infty\},$$

which is a Banach space when endowed with the norm $\|\cdot\|_\alpha^A$. For convenience we further write

$$\mathbb{X}_0^A := \mathbb{X},\ \ \|\mathbb{X}\|_0^A := \|x\|,\ \ \mathbb{X}_1^A := D(A)$$

and $\|x\|_1^A := \|(\omega - A)x\|$. Moreover, let $\widehat{\mathbb{X}}^A := \overline{D(A)}$ of $\mathbb{X}$. In particular, we will adopt the following continuous embedding

$$D(A) \hookrightarrow \mathbb{X}_\beta^A \hookrightarrow D((\omega - A)^\alpha) \hookrightarrow \mathbb{X}_\alpha^A \hookrightarrow \widehat{\mathbb{X}}^A \subset \mathbb{X}, \tag{2.1}$$

for all $0 < \alpha < \beta < 1$, where the fractional powers are defined in the usual way.

In general, $D(A)$ is not dense in the spaces $\mathbb{X}_\alpha^A$ and $\mathbb{X}$. However, we have the following continuous injection:

$$\mathbb{X}_\beta^A \hookrightarrow \overline{D(A)}^{\|\cdot\|_\alpha^A} \tag{2.2}$$

for $0 < \alpha < \beta < 1$.

Definition 2.7 *[13] Given the family of linear operators* $A(t)$ *for* $t \in \mathbb{R}$*, satisfying (H1), we set*

$$\mathbb{X}_\alpha^t := \mathbb{X}_\alpha^{A(t)},\ \ \ \widehat{\mathbb{X}}^t := \widehat{\mathbb{X}}^{A(t)}$$

for $0 \leq \alpha \leq 1$ *and* $t \in \mathbb{R}$*, with the corresponding norms. Then the embedding in Eq. (2.1) holds with constants independent of* $t \in \mathbb{R}$*.*

These interpolation spaces are of class $\mathcal{J}_\alpha$ [18, Definition 1.1.1] and it can be shown that

$$\|y\|_\alpha^t \leq \mathscr{K}^\alpha \mathscr{L}^{1-\alpha} \|y\|^{1-\alpha} \|A(t)y\|^\alpha,\ \ y \in D(A(t)),$$

where $\mathscr{K}$, $\mathscr{L}$ are the constants appearing in (H1).

Lemma 2.6 *[4] For* $x \in \mathbb{X}$*,* $0 \leq \alpha \leq 1$*, the following hold:*
(i) There is a constant $c(\alpha)$*, such that*

$$\|U(t,s)P(s)x\|_\alpha^t \leq c(\alpha) e^{-(\delta/2)(t-s)} (t-s)^{-\alpha} \|x\|,\ \ t > s. \tag{2.3}$$

(ii) There is a constant $m(\alpha)$*, such that*

$$\|\widetilde{U}_Q(s,t)Q(s)x\|_\alpha^s \leq m(\alpha) e^{-\delta(t-s)} \|x\|,\ \ t \leq s. \tag{2.4}$$

Throughout this manuscript, we assume that the function $\mathbb{R} \times \mathbb{R} \mapsto \mathbb{X}$, $(t, s) \mapsto U(t, s)x \in bAA(\mathbb{R} \times \mathbb{R}, \mathbb{X})$ uniformly for x in any bounded subset of $\mathbb{X}$. For the problem (1.1), we list the following assumptions:
(H2) There exists $0 \leq \alpha < \beta < 1$ such that

$$\mathbb{X}_\alpha^t = \mathbb{X}_\alpha$$

for all $t \in \mathbb{R}$, with uniform equivalent norm.

If $0 \leq \alpha < \beta < 1$, then we let k_1 be the bound of the embedding $\mathbb{X}_\alpha \hookrightarrow \mathbb{X}$, that is

$$\|u\| \leq k_1 \|u\|_\alpha \text{ for } u \in \mathbb{X}_\alpha.$$

(H3) Let $0 \leq \alpha < \beta < 1$ and the function $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ belongs to $PAA(\mathbb{R}, \mathbb{X}, \mu)$. Moreover, the function f is uniformly Lipschitz with respect to the second argument in the following sense: there exists $K > 0$ such that

$$\|f(t, u) - f(t, v)\| \leq K \|u - v\|$$

for all $u, v \in \mathbb{X}$ and $t \in \mathbb{R}$.

3 Main results

In the sequel, we suppose that there exist two real numbers α, β such that $0 < \alpha < \beta < 1$ with

$$2\beta > \alpha + 1.$$

Moreover, we denote by Γ_1 and Γ_2 the nonlinear integral operators defined by

$$(\Gamma_1 u)(t) := \int_{-\infty}^{t} U(t, s)P(s)f(s, u(s - h))\,\mathrm{d}s,$$

$$(\Gamma_2 u)(t) := \int_{t}^{\infty} U_Q(t, s)Q(s)f(s, u(s - h))\,\mathrm{d}s,$$

Since the space $\varepsilon(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant, we can easily obtain the following lemma.

Lemma 3.1 *If $u \in PAA(\mathbb{R}, \mathbb{X}, \mu)$ and $h \geq 0$. Then $u(\cdot - h) \in PAA(\mathbb{R}, \mathbb{X}, \mu)$*

Lemma 3.2 *Let $\mu \in \mathcal{M}$, let $u \in PAA(\mathbb{R}, \mathbb{X}_\alpha, \mu)$. Under assumptions (H1)-(H3), the integral operators Γ_1 and Γ_2 defined above map $PAA(\mathbb{R}, \mathbb{X}_\alpha, \mu)$ into itself.*

Proof. Let $u \in PAA(\mathbb{R}, \mathbb{X}_\alpha, \mu)$. Setting $p(t) = f(t, u(t - h))$ and by Theorem 2.1 and Lemma 3.1, it follows that $p \in PAA(\mathbb{R}, \mathbb{X}, \mu)$ for each $u \in PAA(\mathbb{R}, \mathbb{X}_\alpha, \mu)$. Now write $p = \phi + \zeta$ where $\phi \in AA(\mathbb{R}, \mathbb{X})$ and $\zeta \in \varepsilon(\mathbb{R}, \mathbb{X}, \mu)$. Thus $\Gamma_1 u$ can be rewritten as

$$(\Gamma_1(u)(t)) := \int_{-\infty}^{t} U(t, s)P(s)\phi(s)\,\mathrm{d}s + \int_{-\infty}^{t} U(t, s)P(s)\zeta(s)\,\mathrm{d}s.$$

Set $\Phi(t) = \int_{-\infty}^{t} U(t,s)P(s)\phi(s)\,\mathrm{d}s$ and $\Psi(t) = \int_{-\infty}^{t} U(t,s)P(s)\zeta(s)\,\mathrm{d}s$ for each $t \in \mathbb{R}$.
Now, we shall show that $\Phi \in AA(\mathbb{R}, \mathbb{X}_\alpha)$. Let us take a sequence $(s'_n)_{n\in\mathbb{N}}$, since $\phi \in AA(\mathbb{R}, \mathbb{X})$, there is a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$\|\phi(t + s_n - s_m) - \phi(t)\| \le \epsilon v \text{ for each } t \in \mathbb{R},$$

where $v = \delta^{1-\alpha}/c(\alpha)2^{1-\alpha}\Gamma(1-\alpha)$ with Γ being the classical Γ function.

Furthermore,

$$\begin{aligned}
&\Phi(t + s_n - s_m) - \Phi(t)\\
&= \int_{-\infty}^{t+s_n-s_m} U(t+s_n-s_m, s)P(s)\phi(s)\,\mathrm{d}s - \int_{-\infty}^{t} U(t,s)P(s)\phi(s)\,\mathrm{d}s\\
&= \int_{-\infty}^{t} U(t+s_n-s_m, s+s_n-s_m)P(s+s_n-s_m)\phi(s+s_n-s_m)\,\mathrm{d}s - \int_{-\infty}^{t} U(t,s)P(s)\phi(s)\,\mathrm{d}s\\
&= \int_{-\infty}^{t} U(t+s_n-s_m, s+s_n-s_m)P(s+s_n-s_m)\phi(s+s_n-s_m)\,\mathrm{d}s\\
&\quad - \int_{-\infty}^{t} U(t+s_n-s_m, s+s_n-s_m)P(s+s_n-s_m)\phi(s)\,\mathrm{d}s\\
&\quad + \int_{-\infty}^{t} U(t+s_n-s_m, s+s_n-s_m)P(s+s_n-s_m)\phi(s)\,\mathrm{d}s - \int_{-\infty}^{t} U(t,s)P(s)\phi(s)\,\mathrm{d}s\\
&= \int_{-\infty}^{t} U(t+s_n-s_m, s+s_n-s_m)P(s+s_n-s_m)(\phi(s+s_n-s_m) - \phi(s))\,\mathrm{d}s\\
&\quad + \int_{-\infty}^{t} (U(t+s_n-s_m, s+s_n-s_m)P(s+s_n-s_m) - U(t,s)P(s))\phi(s)\,\mathrm{d}s.
\end{aligned}$$

Using [5, 19] it follows that

$$\left\|\int_{-\infty}^{t} (U(t+s_n-s_m, s+s_n-s_m)P(s+s_n-s_m) - U(t,s)P(s))\phi(s)\,\mathrm{d}s\right\|_\alpha \le \frac{2\|\phi\|_\infty}{\delta}\epsilon.$$

Similarly, using Eq. (2.3), it follows that

$$\left\|\int_{-\infty}^{t} U(t+s_n-s_m, s+s_n-s_m)P(s+s_n-s_m)(\phi(s+s_n-s_m) - \phi(s))\,\mathrm{d}s\right\|_\alpha \le \epsilon.$$

Therefore,

$$\|\Phi(t+s_n-s_m) - \Phi(t)\|_\alpha \le \left(1 + \frac{2\|\phi\|_\infty}{\delta}\right)\epsilon \text{ for each } t \in \mathbb{R}$$

and hence, $\Phi \in AA(\mathbb{R}, \mathbb{X}_\alpha)$.

To complete the proof for Γ_1, we have to show that $\Psi(t) \in \varepsilon(\mathbb{R}, \mathbb{X}_\alpha, \mu)$. We have

$$\begin{aligned}
&\frac{1}{\mu([-r,r])} \int_{[-r,r]} \|\Psi(t)\|_\alpha \, \mathrm{d}\mu(t) \\
&= \frac{1}{\mu([-r,r])} \int_{[-r,r]} \| \int_{-\infty}^{t} U(t,s)P(s)\zeta(s) \ \mathrm{d}s\|_\alpha \, \mathrm{d}\mu(t) \\
&\leq \frac{1}{\mu([-r,r])} \int_{[-r,r]} \int_{-\infty}^{t} \|U(t,s)P(s)\zeta(s)\|_\alpha \ \mathrm{d}s \, \mathrm{d}\mu(t) \\
&\leq \frac{1}{\mu([-r,r])} \int_{[-r,r]} \int_{-\infty}^{t} c(\alpha) e^{-(\delta/2)(t-s)} (t-s)^{-\alpha} \|\zeta(s)\| \ \mathrm{d}s \, \mathrm{d}\mu(t) \\
&\leq c(\alpha) \int_{0}^{\infty} s^{-\alpha} e^{-(\delta/2)s} \left(\frac{1}{\mu([-r,r])} \int_{[-r,r]} \|\zeta(t-s)\| \ \mathrm{d}\mu(t) \right) \mathrm{d}s.
\end{aligned}$$

By the fact that the space $\varepsilon(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant, it follows that $t \mapsto \zeta(t-s)$ belongs to $\varepsilon(\mathbb{R}, \mathbb{X}, \mu)$ for each $s \in \mathbb{R}$ and hence

$$\lim_{r \to +\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \|\zeta(t-s)\| \, \mathrm{d}\mu(t) = 0.$$

One completes the proof by using the well-known Lebesgue dominated convergence theorem and the fact $\lim_{r \to +\infty} c(\alpha) \int_0^\infty s^{-\alpha} e^{-(\delta/2)s} \left(\frac{1}{\mu([-r,r])} \int_{[-r,r]} \|\zeta(t-s)\| \ \mathrm{d}\mu(t) \right) ds = 0$. The proof is now completed. $\square$

The proof for $\Gamma_2 u$ is similar to that of $\Gamma_1 u$. However one makes use of Eq. (2.4) rather than Eq. (2.3).

The rest of this section is devoted to the existence of μ-pseudo almost automorphic solutions to the Eq. (1.1).

Definition 3.1 *Let $\alpha \in (0,1)$. A continuous function $u : \mathbb{R} \to \mathbb{X}_\alpha$ is said to be a mild solution to Eq. (1.1) provided that the function $s \to U(t,s)P(s)f(s, u(s-h))$ is integrable on (s,t), $s \to U(t,s)Q(s)f(s,u(s-h))$ is integrable on (t,s) and*

$$u(t) = U(t,s)u(s) + \int_s^t U(t,s)P(s)f(s,u(s-h))\,\mathrm{d}s - \int_t^s U(t,s)Q(s)f(s,u(s-h))\,\mathrm{d}s$$

for each $t \geq s$ and for all t, $s \in \mathbb{R}$.

Theorem 3.1 *Let $\mu \in \mathcal{M}$. Under Assumptions (H1)-(H3), the evolution equation (1.1) has a unique μ-pseudo almost automorphic mild solution whenever K is small enough.*

Proof. Consider the operator $\Lambda : PAA(\mathbb{R}, \mathbb{X}_\alpha, \mu) \to PAA(\mathbb{R}, \mathbb{X}_\alpha, \mu)$ such that

$$\Lambda u(t) := \int_{-\infty}^{t} U(t,s)P(s)f(s,u(s-h))\,\mathrm{d}s - \int_t^\infty U(t,s)Q(s)f(s,u(s-h))\,\mathrm{d}s.$$

As we have previously seen, for every $u \in PAA(\mathbb{R}, \mathbb{X}_\alpha, \mu)$, $f(\cdot, u(\cdot - h)) \in PAA(\mathbb{R}, \mathbb{X}_\alpha, \mu)$. In view of Lemma 3.2, it follows that Λ maps $PAA(\mathbb{R}, \mathbb{X}_\alpha, \mu)$ into itself. To complete the proof one

has to show that Λ has a unique fixed point.
Let $v, w \in PAA(\mathbb{R}, \mathbb{X}_\alpha, \mu)$

$$\begin{aligned}
&\|\Gamma_1(v)(t) - \Gamma_1(w)(t)\|_\alpha \\
&\leq \int_{-\infty}^{t} \|U(t,s)P(s)[f(s,v(s-h)) - f(s,w(s-h))]\|_\alpha \, \mathrm{d}s \\
&\leq \int_{-\infty}^{t} c(\alpha)(t-s)^{-\alpha} e^{-\frac{\delta}{2}(t-s)} \|f(s,v(s-h)) - f(s,w(s-h))\| \, \mathrm{d}s \\
&\leq k_1 c(\alpha) K \int_{-\infty}^{t} (t-s)^{-\alpha} e^{-\frac{\delta}{2}(t-s)} \|v-w\|_\alpha \, \mathrm{d}s \\
&\leq k_1 c(\alpha) K (2\delta^{-1})^{1-\alpha} \Gamma(1-\alpha) \|v-w\|_{\alpha,\infty}
\end{aligned}$$

and

$$\begin{aligned}
&\|\Gamma_2(v)(t) - \Gamma_2(w)(t)\|_\alpha \\
&\leq \int_{t}^{\infty} \|U_Q(t,s)Q(s)[f(s,v(s-h)) - f(s,w(s-h))]\|_\alpha \, \mathrm{d}s \\
&\leq \int_{t}^{\infty} m(\alpha) e^{\delta(t-s)} \|f(s,v(s-h)) - f(s,w(s-h))\| \, \mathrm{d}s \\
&\leq k_1 m(\alpha) K \int_{t}^{\infty} e^{\delta(t-s)} \|v-w\|_\alpha \, \mathrm{d}s \\
&\leq k_1 m(\alpha) K \delta^{-1} \|v-w\|_{\alpha,\infty}.
\end{aligned}$$

Combining previous inequalities it follows that

$$\|\Lambda v - \Lambda w\|_{\alpha,\infty} \leq K\Theta \|v-w\|_{\alpha,\infty},$$

where

$$\Theta := k_1 c(\alpha)(2\delta^{-1})^{1-\alpha}\Gamma(1-\alpha) + k_1 m(\alpha)\delta^{-1}.$$

Therefore, if K is small enough, that is, $K < \Theta^{-1}$, then Eq. (1.1) has a unique solution, which obviously is its only μ-pseudo almost automorphic mild solution. □

From [3], we have the following results.

Remark 3.1 *We consider a locally bounded function $\mathcal{L} : \mathbb{X}_\alpha \times \mathbb{X}_\alpha \to [0,\infty)$ such that for every $r \geq 0$ there is a constant $k(r) \geq 0$ such that $\mathcal{L}(x,y) \leq k(r)$, for all x, $y \in \mathbb{X}_\alpha$ with $\|x\|_\alpha \leq r$ and $\|y\|_\alpha \leq r$.*

Corollary 3.1 *Let $\mu \in \mathcal{M}$. Let also $f = g + p \in PAA(\mathbb{R}, \mathbb{X}_\alpha, \mu)$, assume that there is a locally bounded function $\mathcal{L} : \mathbb{X}_\alpha \times \mathbb{X}_\alpha \to [0,\infty)$ such that for every x, $y \in \mathbb{X}_\alpha$ we have*

$$\|f(t,x) - f(t,y)\| \leq \mathcal{L}(x,y)(1 + \|x\|_\alpha^{l-1} + \|y\|_\alpha^{l-1})\|x-y\|_\alpha, \ \ (t \in \mathbb{R}),$$
$$\|g(t,x) - g(t,y)\| \leq \mathcal{L}(x,y)(1 + \|x\|_\alpha^{l-1} + \|y\|_\alpha^{l-1})\|x-y\|_\alpha, \ \ (t \in \mathbb{R}),$$

where $l \geq 1$. If there is $R \geq 0$ such that

$$\Theta = K(R)\left(c(\alpha)(2\delta^{-1})^{1-\alpha}\Gamma(1-\alpha) + m(\alpha)\delta^{-1}\right) < 1,$$

where $K(R) := k(R)(1 + 2R^{l-1})$, with $k(R)$ as in Remark 3.1, and $c(\alpha)$ and $m(\alpha)$ are the constants given in Lemma 2.6. Then Eq. (1.1) has a unique μ-pseudo almost automorphic mild solution.

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