Journal of Nonlinear Evolution Equations and Applications ISSN 2161-3680
Volume 2014, Number 1, pp. 1–12 (August 2014) http://www.jneea.com

ULAM STABILITY FOR IMPULSIVE DISCONTINUOUS PARTIAL FRACTIONAL DIFFERENTIAL EQUATIONS IN BANACH ALGEBRAS

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Received on September 6, 2013

Accepted on January 17, 2014

Communicated by Gisèle Mophou

Abstract. In this paper, we investigate some existence and Ulam's type stability concepts of fixed point inclusions for a class of partial discontinuous fractional order differential equations with impulses in Banach algebras.

Keywords: Partial fractional differential equation, left-sided mixed Riemann–Liouville integral, Caputo fractional-order derivative, fixed point equation, Banach algebra, impulse, Ulam–Hyers stability.

2010 Mathematics Subject Classification: 26A33, 34G20, 34A40, 45N05, 47H10.

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1 Introduction

The fractional calculus represents a powerful tool in applied mathematics to study a myriad of problems from different fields of science and engineering, with many break-through results found in mathematical physics, finance, hydrology, biophysics, thermodynamics, control theory, statistical mechanics, astrophysics, cosmology and bioengineering [9, 19, 23]. There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Abbas *et al.* [4], Kilbas *et al.* [15], Miller and Ross [16], the papers of Abbas *et al.* [1, 2, 3, 5, 6], Vityuk and Golushkov [25], and the references therein.

The stability of functional equations was originally raised by Ulam in 1940 in a talk given at Wisconsin University (for more details see [24]). The first answer to Ulam's question was given by Hyers in 1941 in the case of Banach spaces in [10]. Thereafter, this type of stability is called the Ulam–Hyers stability. In 1978, Rassias [20] provided a remarkable generalization of the Ulam–Hyers stability of mappings by considering variables. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus, the stability question of functional equations is how do the solutions of the inequality differ from those of the given functional equation? Considerable attention has been given to the study of the Ulam–Hyers and Ulam–Hyers–Rassias stability of all kinds of functional equations; one can see the monographs of [11, 12]. Bota-Boriceanu and Petrusel [7], Petru *et al.* [17, 18], and Rus [21, 22] discussed the Ulam–Hyers stability for operatorial equations and inclusions. Castro and Ramos [8], and Jung [14] considered the Hyers–Ulam–Rassias stability for a class of Volterra integral equations. Ulam stability for fractional differential equations with Caputo derivative are proposed by Wang *et al.* [26, 27]. Some stability results for fractional integral equation are obtained by Wei *et al.* [28]. More details from historical point of view, and recent developments of such stabilities are reported in [13, 21, 28].

In this article, we discuss the Ulam–Hyers–Rassias stability for the following fractional partial impulsive discontinuous differential equations of the form

$$\begin{cases} {}^cD^r_{\theta_k}\left(\dfrac{u(x,y)}{f(x,y,u(x,y))}\right) = g(x,y,u(x,y));\ (x,y)\in J_k,\ k=0,\dots,m, \\ u(x_k^+,y) = u(x_k^-,y) + I_k(u(x_k^-,y));\ y\in[0,b],\ k=1,\dots,m, \\ u(x,0)=\varphi(x);\ x\in[0,a],\ u(0,y)=\psi(y);\ y\in[0,b], \end{cases} \tag{1.1}$$

where $a,b>0,\ J_0=[0,x_1]\times[0,b],\ J_k:=(x_k,x_{k+1}]\times[0,b];\ k=1,\dots,m,\ \theta_k=(x_k,0);\ k=0,\dots,m,\ 0=x_0<x_1<\cdots<x_m<x_{m+1}=a,\ {}^cD^r_{\theta_k}$ is the fractional Caputo derivative of order $r=(r_1,r_2)\in(0,1]\times(0,1],\ f\colon J\times\mathbb{R}\to\mathbb{R}^*,\ g\colon J\times\mathbb{R}\to\mathbb{R}$ are given continuous functions, $J=[0,a]\times[0,b],\ \mathbb{R}^*=\mathbb{R}\backslash\{0\},\ I_k\colon\mathbb{R}\to\mathbb{R};\ k=1,\dots,m$ are given functions satisfying suitable conditions and $\varphi\colon[0,a]\to\mathbb{R},\ \psi\colon[0,b]\to\mathbb{R}$ are given absolutely continuous functions with $\varphi(0)=\psi(0)$. Here $u(x_k^+,y)$ and $u(x_k^-,y)$ denote the right and left limits of $u(x,y)$ at $x=x_k$, respectively.

2 Preliminaries

Denote $L^1(J)$ the space of Lebesgue-integrable functions $u\colon J\to\mathbb{R}$ with the norm

$$\|u\|_{L^1}=\int_0^a\int_0^b|u(x,y)|\,\mathrm{d}y\,\mathrm{d}x.$$

As usual, by $AC(J)$ we denote the space of absolutely continuous functions from J into $\mathbb{R}$, and $\mathcal{C} := C(J)$ is the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm $\|.\|_\infty$ defined by

$$\|u\|_\infty = \sup_{(x,y)\in J} |u(x,y)|.$$

In all what follows consider the Banach space

$$\mathcal{PC} := \left\{ u\colon J \to \mathbb{R} : \begin{array}{c} u \in C(J_k) \text{ for } k = 0,1,\ldots,m, \text{ and there exist } u(x_k^-,y) \text{ and } u(x_k^+,y), \\ k = 1,\ldots,m, \text{ with } u(x_k^-,y) = u(x_k,y) \text{ for each } y \in [0,b] \end{array} \right\}$$

with the norm

$$\|u\|_{\mathcal{PC}} = \sup_{(x,y)\in J} |u(x,y)|.$$

Define a multiplication " $\cdot$ " by

$$(u \cdot v)(x,y) = u(x,y)v(x,y) \text{ for each } (x,y) \in J.$$

Then $\mathcal{PC}$ is a Banach algebra with the above norm and multiplication.

Definition 2.1 *A function $\gamma\colon J \times \mathbb{R} \to \mathbb{R}$ is said to be* Carathéodory *if*

(i) *the function $(x,y) \mapsto \gamma(x,y,u)$ is measurable for each $u \in \mathbb{R}$;*

(ii) *the function $u \mapsto \gamma(x,y,u)$ is continuous for almost each $(x,y) \in J$.*

Now, we introduce notations and definitions concerning partial fractional calculus theory.

Definition 2.2 ([25]) *Let $\theta = (0,0)$, $r_1, r_2 \in (0,\infty)$ and $r = (r_1,r_2)$. For $f \in L^1(J)$, the expression*

$$(I_\theta^r f)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t)\,\mathrm{d}t\,\mathrm{d}s,$$

is called the left-sided mixed Riemann–Liouville integral of order r, where $\Gamma(.)$ is the (Euler's) Gamma function defined by $\Gamma(\xi) = \int_0^\infty t^{\xi-1}e^{-t}\,\mathrm{d}t$ for $\xi > 0$.

In particular,

$$(I_\theta^\theta f)(x,y) = f(x,y), \quad (I_\theta^\sigma f)(x,y) = \int_0^x \int_0^y f(s,t)\,\mathrm{d}t\,\mathrm{d}s \quad \text{for almost all } (x,y) \in J,$$

where $\sigma = (1,1)$.

For instance, $I_\theta^r f$ exists for all $r_1, r_2 \in (0,\infty)$, when $f \in L^1(J)$. Note also that when $u \in \mathcal{C}$, then $(I_\theta^r f) \in \mathcal{C}$. Moreover

$$(I_\theta^r f)(x,0) = (I_\theta^r f)(0,y) = 0;\; x \in [0,a],\; y \in [0,b].$$

Example 2.3 *Let $\lambda, \omega \in (-1,0) \cup (0,\infty)$ and $r = (r_1,r_2) \in (0,\infty) \times (0,\infty)$, then*

$$I_\theta^r x^\lambda y^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)} x^{\lambda+r_1} y^{\omega+r_2} \quad \textit{for almost all } (x,y) \in J.$$

By $1-r$ we mean $(1-r_1, 1-r_2) \in [0,1) \times [0,1)$. Denote by $D^2_{xy} := \frac{\partial^2}{\partial x \partial y}$ the mixed second order partial derivative.

Definition 2.4 ([25]) *Let $r \in (0,1] \times (0,1]$ and $f \in L^1(J)$. The Caputo fractional-order derivative of order r of f is defined by the expression*

$$^cD^r_\theta f(x,y) = (I^{1-r}_\theta D^2_{xy} f)(x,y) = \frac{1}{\Gamma(1-r_1)\Gamma(1-r_2)} \int_0^x \int_0^y \frac{D^2_{st} f(s,t)}{(x-s)^{r_1}(y-t)^{r_2}} \, \mathrm{d}t \, \mathrm{d}s.$$

The case $\sigma = (1,1)$ is included and we have

$$(^cD^\sigma_\theta f)(x,y) = (D^2_{xy} f)(x,y) \quad \text{for almost all } (x,y) \in J.$$

Example 2.5 *Let $\lambda, \omega \in (-1,0) \cup (0,\infty)$ and $r = (r_1, r_2) \in (0,1] \times (0,1]$, then*

$$^cD^r_\theta x^\lambda y^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda-r_1)\Gamma(1+\omega-r_2)} x^{\lambda-r_1} y^{\omega-r_2} \quad \textit{for almost all } (x,y) \in J.$$

Let $a_1 \in [0,a]$, $z = (a_1, 0)$, $J_z = (a_1, a] \times [0,b]$, $r_1, r_2 > 0$ and $r = (r_1, r_2)$. For $u \in L^1(J_z)$, the expression

$$(I^r_z u)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{a_1^+}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} u(s,t) \, \mathrm{d}t \, \mathrm{d}s,$$

is called the left-sided mixed Riemann–Liouville integral of order r of u.

Definition 2.6 ([25]) *For $u \in L^1(J_z)$, where $D^2_{xy} u$ is Lebesque integrable on $[x_k, x_{k+1}] \times [0,b]$, $k = 0, \ldots, m$, the Caputo fractional order derivative of order r of u is defined by the expression*

$$(^cD^r_z f)(x,y) = (I^{1-r}_z D^2_{xy} f)(x,y).$$

Let

$$\mu_k(x,y) = \frac{u(x,0)}{f(x,0,u(x,0))} + \frac{u(x_k^+, y)}{f(x_k^+, y, u(x_k^+, y))} - \frac{u(x_k^+, 0)}{f(x_k^+, 0, u(x_k^+, 0))}; \quad k = 0, \ldots, m.$$

For the existence of solutions for the problem (1.1) we need the following lemmas.

Lemma 2.7 ([1]) *A function $u \in AC(J_k)$, $k = 0, \ldots, m$, is said to be a solution of the differential equation*

$$^cD^r_{\theta_k} \left(\frac{u(x,y)}{f(x,y,u(x,y))} \right) = g(x,y,u(x,y)), \quad (x,y) \in J_k, \tag{2.1}$$

if and only if $u(x,y)$ satisfies

$$u(x,y) = f(x,y,u(x,y)) \Big(\mu_k(x,y) + (I^r_{\theta_k} g)(x,y,u(x,y)) \Big), \quad (x,y) \in J_k. \tag{2.2}$$

Let $\mu := \mu_0$.

Lemma 2.8 ([1]) *A function u is a solution of the fractional integral equations*

$$\begin{cases} u(x,y) = f(x,y,u(x,y))\Big[\mu(x,y) \\ \qquad + \dfrac{1}{\Gamma(r_1)\Gamma(r_2)}\displaystyle\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}g(s,t,u(s,t))\,\mathrm{d}t\,\mathrm{d}s\Big], \text{ if } (x,y)\in J_0; \\ \\ u(x,y) = f(x,y,u(x,y))\Big[\mu(x,y) \\ \qquad + \displaystyle\sum_{i=1}^{k}\left(\frac{I_i(u(x_i^-,y))}{f(x_i^+,y,u(x_i^+,y))} - \frac{I_i(u(x_i^-,0))}{f(x_i^+,0,u(x_i^+,0))}\right) \\ \qquad + \dfrac{1}{\Gamma(r_1)\Gamma(r_2)}\displaystyle\sum_{i=1}^{k}\int_{x_{i-1}}^{x_i}\int_0^y (x_i-s)^{r_1-1}(y-t)^{r_2-1}g(s,t,u(s,t))\,\mathrm{d}t\,\mathrm{d}s \\ \qquad + \displaystyle\int_{x_k}^x\int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}g(s,t,u(s,t))\,\mathrm{d}t\,\mathrm{d}s\Big], \text{ if } (x,y)\in J_k,\ k=1,\dots,m, \end{cases}$$

if and only if u is a solution of the problem (1.1).

Remark 2.9 *By Lemma 2.8, solutions of the problem* (1.1) *are solutions of the fixed point equation $u = N(u)$ where $N\colon \mathcal{PC}\to\mathcal{PC}$ is the operator defined by*

$$\begin{cases} (Nu)(x,y) = f(x,y,u(x,y))\Big[\mu(x,y) \\ \qquad + \displaystyle\int_0^x\int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}g(s,t,u(s,t))\,\mathrm{d}t\,\mathrm{d}s\Big], \text{ if } (x,y)\in J_0; \\ \\ (Nu)(x,y) = f(x,y,u(x,y))\Big[\mu(x,y) \\ \qquad + \displaystyle\sum_{i=1}^{k}\left(\frac{I_i(u(x_i^-,y))}{f(x_i^+,y,u(x_i^+,y))} - \frac{I_i(u(x_i^-,0))}{f(x_i^+,0,u(x_i^+,0))}\right) \\ \qquad + \dfrac{1}{\Gamma(r_1)\Gamma(r_2)}\displaystyle\sum_{i=1}^{k}\int_{x_{i-1}}^{x_i}\int_0^y (x_i-s)^{r_1-1}(y-t)^{r_2-1}g(s,t,u(s,t))\,\mathrm{d}t\,\mathrm{d}s \\ \qquad + \displaystyle\int_{x_k}^x\int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}g(s,t,u(s,t))\,\mathrm{d}t\,\mathrm{d}s\Big], \text{ if } (x,y)\in J_k,\ k=1,\dots,m. \end{cases}$$

Let us give the definition of Ulam–Hyers stability of a fixed point equation due to Rus.

Definition 2.10 ([22]) *Let (X,d) be a metric space and $A\colon X\to X$ be an operator. The fixed point equation $x = A(x)$ is said to be Ulam–Hyers stable if there exists a real number $c_A > 0$ such that: for each real number $\epsilon > 0$ and each solution y^* of the inequality $d(y,A(y)) \le \epsilon$, there exists a solution x^* of the equation $x = A(x)$ such that*

$$d(y^*,x^*) \le \epsilon c_A\,.$$

From the above definition, we shall give four types of Ulam stability of the fixed point equation $u = N(u)$. Let ϵ be a positive real number and let $\Phi\colon J \to [0,\infty)$ be a continuous function.

Definition 2.11 *The fixed point equation $u = N(u)$ is said to be Ulam–Hyers stable if there exists a real number $c_N > 0$ such that for each $\epsilon > 0$ and for each solution $u \in \mathcal{PC}$ of the inequality $|u(x,y) - (Nu)(x,y)| \leq \epsilon$, $(x,y) \in J$, there exists a solution $v \in \mathcal{PC}$ of the equation $u = N(u)$ with*

$$|u(x,y) - v(x,y)| \leq \epsilon c_N, \quad (x,y) \in J.$$

Definition 2.12 *The fixed point equation $u = N(u)$ is said to be generalized Ulam–Hyers stable if there exists $\Theta_N \in C([0,\infty),[0,\infty)),\ \Theta_N(0) = 0$ such that for each $\epsilon > 0$ and for each solution $u \in \mathcal{PC}$ of the inequality $|u(x,y) - (Nu)(x,y)| \leq \epsilon$, $(x,y) \in J$, there exists a solution $v \in \mathcal{PC}$ of the equation $u = N(u)$ with*

$$|u(x,y) - v(x,y)| \leq \Theta_N(\epsilon), \quad (x,y) \in J.$$

Definition 2.13 *The fixed point equation $u = N(u)$ is said to be Ulam–Hyers–Rassias stable with respect to Φ if there exists a real number $c_{N,\Phi} > 0$ such that for each $\epsilon > 0$ and for each solution $u \in \mathcal{PC}$ of the inequality $|u(x,y) - (Nu)(x,y)| \leq \epsilon\Phi(x,y)$, $(x,y) \in J$, there exists a solution $v \in \mathcal{PC}$ of the equation $u = N(u)$ with*

$$|u(x,y) - v(x,y)| \leq \epsilon c_{N,\Phi}\Phi(x,y), \quad (x,y) \in J.$$

Definition 2.14 *The fixed point equation $u = N(u)$ is said to be generalized Ulam–Hyers–Rassias stable with respect to Φ if there exists a real number $c_{N,\Phi} > 0$ such that for each solution $u \in \mathcal{PC}$ of the inequality $|u(x,y) - (Nu)(x,y)| \leq \Phi(x,y)$, $(x,y) \in J$, there exists a solution $v \in \mathcal{PC}$ of the equation $u = N(u)$ with*

$$|u(x,y) - v(x,y)| \leq c_{N,\Phi}\Phi(x,y), \quad (x,y) \in J.$$

Remark 2.15 *It is clear that*

(i) *Definition 2.11 $\Rightarrow$ Definition 2.12*;

(ii) *Definition 2.13 $\Rightarrow$ Definition 2.14*;

(iii) *Definition 2.13 for $\Phi(x,y) = 1 \ \Rightarrow$ Definition 2.11.*

3 Ulam–Hyers–Rassias Stability Result

In this section, we present the main result for the Ulam stability of the problem (1.1).

Definition 3.1 *A function $w \in \mathcal{PC}$ such that its mixed derivative D^2_{xy} exists and is integrable on J_k, $k = 0,\dots,m$, is said to be a solution of the problem* (1.1) *if*

(i) *the function* $(x,y) \mapsto \dfrac{w(x,y)}{f(x,y,w(x,y))}$ *is absolutely continuous, and*

(ii) w *satisfies* ${}^cD^r_{\theta_k}\big(\frac{w(x,y)}{f(x,y,w(x,y))}\big) = g(x,y,w(x,y))$ *on* J_k *and the conditions*

$$\begin{cases} w(x_k^+,y) = w(x_k^-,y) + I_k(w(x_k^-,y));\ y\in[0,b],\ k=1,\dots,m, \\ w(x,0)=\varphi(x);\ x\in[0,a],\ w(0,y)=\psi(y);\ y\in[0,b], \end{cases}$$

are satisfied.

Let us start by giving conditions for the Ulam–Hyers stability of problem (1.1).

Theorem 3.2 *Assume that the following hypotheses hold:*

(H_1) *there exists a strictly positive function* $\alpha \in \mathcal{C}$ *such that*

$$|f(x,y,u) - f(x,y,\overline{u})| \le \alpha(x,y)|u-\overline{u}| \quad \textit{for all } (x,y)\in J \textit{ and } u,\overline{u}\in\mathbb{R};$$

(H_2) *the function* g *is Carathéodory, and there exists* $h \in L^\infty(J,\mathbb{R}_+)$ *such that*

$$|g(x,y,u)| \le h(x,y); \quad \textit{a.e. } (x,y)\in J, \textit{ for all } u\in\mathbb{R};$$

(H_3) *there exists a positive function* $\beta\in\mathcal{C}$ *such that*

$$\left|\frac{I_k(u)}{f(x,y,u)}\right| \le \beta(x,y) \quad \textit{for all } (x,y)\in J \textit{ and } u\in\mathbb{R}.$$

If

$$L := \|\alpha\|_\infty\left[\|\mu\|_\infty + 2m\|\beta\|_\infty + \frac{2a^{r_1}b^{r_2}\|h\|_{L^\infty}}{\Gamma(1+r_1)\Gamma(1+r_2)}\right] < 1, \tag{3.1}$$

then the problem (1.1) *has at least one solution on* J*. Moreover, if the following hypothesis*:

(H_4) *there exists* $\lambda_\Phi > 0$ *such that, for each* $(x,y)\in J$ *and* $u\in\mathbb{R}$ *we have*

$$|f(x,y,u)| \le \lambda_\Phi \Phi(x,y),$$

holds, then the fixed point equation $u = N(u)$ *is generalized Ulam–Hyers–Rassias stable.*

Proof. Let N be the operator defined in Remark 2.9. From [1, Theorem 4.1], we have that the problem (1.1) has at least one solution on J. Now, we prove the generalized Ulam–Hyers–Rassias stability of the operator N. Let $u\in\mathcal{PC}$ be a solution of the inequality $|u-N(u)| \le \Phi(x,y)$ on J, and let v be a solution of the fixed point equation $u = N(u)$. Then we have

$$\begin{cases} v(x,y) = f(x,y,v(x,y))\Big[\mu(x,y) \\ \quad + \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} g(s,t,v(s,t))\,\mathrm{d}t\,\mathrm{d}s\Big], \text{ if } (x,y) \in J_0; \\ \\ v(x,y) = f(x,y,v(x,y))\Big[\mu(x,y) \\ \quad + \sum_{i=1}^{k} \left(\frac{I_i(v(x_i^-,y))}{f(x_i^+,y,v(x_i^+,y))} - \frac{I_i(v(x_i^-,0))}{f(x_i^+,0,v(x_i^+,0))} \right) \\ \quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} \int_0^y (x_i-s)^{r_1-1}(y-t)^{r_2-1} g(s,t,v(s,t))\,\mathrm{d}t\,\mathrm{d}s \\ \quad + \int_{x_k}^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} g(s,t,v(s,t))\ \mathrm{d}t\mathrm{d}s\Big], \text{ if } (x,y) \in J_k,\, k = 1,\dots,m. \end{cases}$$

Then, for each $(x,y) \in J$, it follows that

$$\begin{aligned} |u(x,y) - v(x,y)| &= |u(x,y) - (Nv)(x,y)| \\ &\le |u(x,y) - (Nu)(x,y)| + |(Nu)(x,y) - (Nv)(x,y)| \\ &\le \Phi(x,y) + |(Nu)(x,y) - (Nv)(x,y)|. \end{aligned}$$

Thus, for each $(x,y) \in J_0$, we have

$$\begin{aligned} |u(x,y) - v(x,y)| \le\ & \Phi(x,y) + |f(x,y,u(x,y)) - f(x,y,v(x,y))| \\ & \times \left| \mu(x,y) + \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} g(s,t,u(s,t))\,\mathrm{d}t\,\mathrm{d}s \right| \\ & + |f(x,y,v(x,y))| \\ & \times \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} |g(s,t,u(s,t)) - g(s,t,v(s,t))|\,\mathrm{d}t\,\mathrm{d}s \\ \le\ & \Phi(x,y) + \|\alpha\|_\infty |u(x,y) - v(x,y)| \\ & \times \left[\|\mu\|_\infty + \frac{a^{r_1} b^{r_2} \|h\|_{L^\infty}}{\Gamma(1+r_1)\Gamma(1+r_2)} \right] \\ & + \frac{2a^{r_1} b^{r_2} \|h\|_{L^\infty}}{\Gamma(1+r_1)\Gamma(1+r_2)} \lambda_\Phi \Phi(x,y) \\ \le\ & \Phi(x,y) + L|u(x,y) - v(x,y)| + \frac{2L}{\|\alpha\|_\infty} \lambda_\Phi \Phi(x,y), \end{aligned}$$

and for each $(x,y) \in J_k,\, k = 1,\dots,m$, we get

$$\begin{aligned} |u(x,y) - v(x,y)| \le\ & \Phi(x,y) + |f(x,y,u(x,y)) - f(x,y,v(x,y))| \\ & \times \left| \mu(x,y) + \sum_{i=1}^{k} \left(\frac{I_i(u(x_i^-,y))}{f(x_i^+,y,u(x_i^+,y))} - \frac{I_i(u(x_i^-,0))}{f(x_i^+,0,u(x_i^+,0))} \right) \right. \end{aligned}$$

$$
\begin{aligned}
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} \int_0^y (x_i - s)^{r_1-1}(y-t)^{r_2-1} g(s,t,u(s,t))\,\mathrm{d}t\,\mathrm{d}s \\
&\quad + \int_{x_k}^{x} \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} g(s,t,u(s,t))\,\mathrm{d}t\,\mathrm{d}s \Bigg| \\
&+ |f(x,y,v(x,y))| \\
&\times \Bigg[\sum_{i=1}^{k} \left| \frac{I_i(u(x_i^-,y))}{f(x_i^+,y,u(x_i^+,y))} - \frac{I_i(v(x_i^-,y))}{f(x_i^+,y,v(x_i^+,y))} \right| \\
&\quad + \sum_{i=1}^{k} \left| \frac{I_i(u(x_i^-,0))}{f(x_i^+,0,u(x_i^+,0))} - \frac{I_i(v(x_i^-,0))}{f(x_i^+,0,v(x_i^+,0))} \right| \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} \int_0^y (x_i - s)^{r_1-1}(y-t)^{r_2-1} \\
&\quad \times |g(s,t,u(s,t)) - g(s,t,v(s,t))| \mathrm{d}t\,\mathrm{d}s \\
&\quad + \int_{x_k}^{x} \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} |g(s,t,u(s,t)) - g(s,t,v(s,t))| \mathrm{d}t\,\mathrm{d}s \Bigg] \\
&\leq \Phi(x,y) + L|u(x,y) - v(x,y)| + \frac{2L}{\|\alpha\|_\infty} \lambda_\Phi \Phi(x,y).
\end{aligned}
$$

Hence, by (3.1) for each $(x,y) \in J_k$, $k = 0, \dots, m$, we get

$$
\begin{aligned}
|u(x,y) - v(x,y)| &\leq \frac{1}{1-L}\left(1 + \frac{2L\lambda_\Phi}{\|\alpha\|_\infty}\right)\Phi(x,y) \\
&=: c_{N,\Phi}\Phi(x,y).
\end{aligned}
$$

Consequently, the fixed point equation $u = N(u)$ is generalized Ulam–Hyers–Rassias stable. □

4 More existence and Ulam stability results

Now we present (without proof) some existence and Ulam stability results to the following problem

$$
\begin{cases}
{}^cD_{\theta_k}^r\left(\dfrac{u(x,y)}{f(x,y,u(x,y))}\right) = g(x,y,u(x,y));\ (x,y) \in J := [0,a] \times [0,b], \\
u(x,0) = \varphi(x);\ x \in [0,a],\ \ u(0,y) = \psi(y);\ y \in [0,b],
\end{cases}
\tag{4.1}
$$

where $a, b > 0$, $\theta = (0,0)$, ${}^cD_\theta^r$ is the Caputo's fractional derivative of order $r = (r_1, r_2) \in (0,1] \times (0,1]$, $f\colon J \times \mathbb{R} \to \mathbb{R}^*$, $g\colon J \times \mathbb{R} \to \mathbb{R}$ are given continuous function, and $\varphi \in AC([0,a])$,$\psi \in AC([0,b])$ with $\varphi(0) = \psi(0)$.

Remark 4.1 *Solutions of the problem* (4.1) *are solutions of the fixed point equation* $u = \overline{N}(u)$ *where* $\overline{N}\colon \mathcal{C} \to \mathcal{C}$ *is the operator defined by*

$$
(\overline{N}u)(x,y) = f(x,y,u(x,y))\left[\mu(x,y) + \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} g(s,t,u(s,t))\,\mathrm{d}t\,\mathrm{d}s\right].
$$

Theorem 4.2 *Assume that hypotheses* (H_1) *and* (H_2) *hold. If*

$$\|\alpha\|_\infty\left[\|\mu\|_\infty+\frac{a^{r_1}b^{r_2}\|h\|_{L^\infty}}{\Gamma(1+r_1)\Gamma(1+r_2)}\right]<1, \tag{4.2}$$

then the problem (4.1) *has at least one solution on* J*. Moreover, if the hypothesis* (H_4) *holds, then the fixed point equation* $u=\overline{N}(u)$ *is generalized Ulam–Hyers–Rassias stable.*

5 An Example

As an application of our results we consider the following problem:

$$\begin{cases} {}^cD^r_{\theta_k}\left(\dfrac{u(x,y)}{f(x,y,u(x,y))}\right)=g(x,y,u(x,y));\ (x,y)\in[0,1]\times[0,1],\ x\neq\frac{1}{2},k=0,1,\\ u\left(\frac{1}{2}^+,y\right)=u\left(\frac{1}{2}^-,y\right)+I_1\left(u\left(\frac{1}{2}^-,y\right)\right);\ y\in[0,1],\\ u(x,0)=\varphi(x),\ u(0,y)=\psi(y);\ x,y\in[0,1], \end{cases} \tag{5.1}$$

where $\theta_1=(\frac{1}{2},0),\ f,g\colon[0,1]\times[0,1]\times\mathbb{R}\to\mathbb{R},\ I_1\colon\mathbb{R}\to\mathbb{R},\ \varphi,\psi\colon[0,1]\to\mathbb{R},$

$$f(x,y,u)=\frac{1}{e^{x+y+10}(1+|u|)},\quad g(x,y,u)=\frac{1}{e^{x+y+8}(1+u^2)},$$

and

$$I_1(u)=\frac{(8+e^{-10})^2}{512e^{10}(1+|u|)^2},$$

$$\varphi(x)=\begin{cases}\frac{x^2}{2}e^{-10}, & \text{if } x\in[0,\frac{1}{2}],\\ x^2e^{-10}, & \text{if } x\in(\frac{1}{2},1],\end{cases}$$

and

$$\psi(y)=ye^{-10}\quad\text{for all } y\in[0,1].$$

We can see that the solutions of the problem (5.1) are solutions of the fixed point equation $u=A(u)$ where $A\colon PC([0,1]\times[0,1],\mathbb{R})\to PC([0,1]\times[0,1],\mathbb{R})$ is the operator defined by

$$\begin{cases} (Au)(x,y)=f(x,y,u(x,y))[\mu(x,y)+I^r_\theta g(x,y,u(x,y))], \quad \text{if } (x,y)\in J_0:=[0,\frac{1}{2}]\times[0,1],\\ \\ (Au)(x,y)=f(x,y,u(x,y))\Big[\mu(x,y)+\\ \qquad +\dfrac{(I_1(u(\frac{1}{2}^-,y))}{f(\frac{1}{2}^+,y,u(\frac{1}{2}^+,y))}-\dfrac{I_1(u(\frac{1}{2}^-,0)))}{f(\frac{1}{2}^+,0,u(\frac{1}{2}^+,0))}\\ \qquad +I^r_\theta g(\frac{1}{2},y,u(\frac{1}{2},y))+I^r_{\theta_1}g(x,y,u(x,y))\Big], \quad \text{if } (x,y)\in J_1:=(\frac{1}{2},1]\times[0,1]. \end{cases}$$

The function f is continuous and satisfies (H_1) with $\alpha(x,y)=\dfrac{1}{e^{x+y+10}}$. Then $\|\alpha\|_\infty=\dfrac{1}{e^{10}}$. Also, the function g satisfies (H_2) with $h(x,y)=\dfrac{1}{e^{x+y+8}}$, and so $\|h\|_{L^\infty}=\dfrac{1}{e^8}$. The condition (H_3)

holds with $\beta(x,y) = \dfrac{81e^{x+y}}{512}$. This gives $\|\beta\|_\infty = \dfrac{81e^2}{512}$. A simple computation gives $\|\mu\|_\infty < 4e$. The condition (3.1) holds with $a = b = 1,\ m = 1$. Indeed, $\Gamma(1+r_i) > \frac{1}{2};\ i = 1,2$. A simple computation shows that

$$
\begin{aligned}
L &= \|\alpha\|_\infty \left[\|\mu\|_\infty + 2m\|\beta\|_\infty + \frac{2a^{r_1}b^{r_2}\|h\|_{L^\infty}}{\Gamma(1+r_1)\Gamma(1+r_2)} \right] \\
&< \frac{1}{e^{10}} \left[4e + \frac{81e^2}{256} + \frac{8}{e^8} \right] \\
&< 1.
\end{aligned}
$$

Finally, we can see that the hypothesis (H_4) is satisfied with $\Phi(x,y) = \dfrac{1}{e^{x+y+8}}$ and $\lambda_\Phi = 1$. Consequently, Theorem 3.2 implies that the problem (5.1) has a solution defined on $[0,1] \times [0,1]$, and the fixed point equation $u = A(u)$ is generalized Ulam–Hyers–Rassias stable.

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