BASICS OF RIGHT NABLA FRACTIONAL CALCULUS ON TIME SCALES

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Received October 18, 2011
Accepted October 23, 2011

Communicated by Maximilian F. Hasler

Abstract. We develop the right nabla fractional calculus on time scales. We introduce the related Riemann-Liouville type fractional integral and Caputo like fractional derivative and prove a fractional Taylor formula with integral remainder.

Keywords: Fractional calculus on time scales.

2010 Mathematics Subject Classification: 26A33, 39A12, 93C70.

1 Background

For the basics of times scales the reader is referred to [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12].

Let $\mathbb{T}$ be a time scale, and $\hat{h}_k : \mathbb{T}^2 \to \mathbb{R}$, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, such that $\forall s, t \in \mathbb{T}$, $\hat{h}_0 (t, s) = 1$,

$$\hat{h}_{k+1} (t, s) = \int_s^t \hat{h}_k (\tau, s) \nabla \tau. \quad (1.1)$$

Here $\hat{h}_k$ are ld-continuous in $t$, and

$$\hat{h}_k^\nabla (t, s) = \hat{h}_{k-1} (t, s), \ k \in \mathbb{N}, \ t \in \mathbb{T}_k,$$

with $\hat{h}_1 (t, s) = t - s, \forall s, t \in \mathbb{T}$.

From [3], we write down Taylor’s formula in terms of nabla polynomials:

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Theorem 1 Assume that $T = T_k$. Let $f \in C^m_{ld}(T, \mathbb{R})$, $m \in \mathbb{N}$, $b, t \in T$. Then

$$f(t) = \sum_{k=0}^{m-1} \hat{h}_k(t,b) f^\nabla_k(b) + \int_b^t \hat{h}_{m-1}(t,\rho(\tau)) f^\nabla_m(\tau) \nabla \tau. \quad (1.2)$$

Call

$$R^b_m(f)(t) := \int_b^t \hat{h}_{m-1}(t,\rho(\tau)) f^\nabla_m(\tau) \nabla \tau = - \int_t^b \hat{h}_{m-1}(t,\rho(\tau)) f^\nabla_m(\tau) \nabla \tau. \quad (1.3)$$

Following [3], we define

$$\hat{g}_0(t,s) = 1,$$

$$\hat{g}_{n+1}(t,s) = \int_s^t \hat{g}_n(\rho(\tau),s) \nabla \tau, \quad n \in \mathbb{N}, s,t \in T. \quad (1.4)$$

Notice here

$$\hat{g}_n^\nabla(t,s) = \hat{g}_n(\rho(t),s), \quad t \in T_k,$$

$$\hat{g}_1(t,s) = t - s, \quad \forall s,t \in T.$$

From [3] we need

Theorem 2 If $T = T_k = T^k$, and $n \in \mathbb{N}_0$, then

$$\hat{h}_n(t,s) = (-1)^n \hat{g}_n(s,t), \quad \forall s,t \in T. \quad (1.5)$$

By Theorem 2 we get

$$R^b_m(f)(t) = (-1)^m \int_t^b \hat{g}_{m-1}(\rho(\tau),t) f^\nabla_m(\tau) \nabla \tau. \quad (1.6)$$

We make

Definition 3 Let $\alpha \geq 0$ real number. We consider the continuous functions

$$\hat{g}_\alpha : T^2 \rightarrow \mathbb{R},$$

such that

$$\hat{g}_0(t,s) = 1,$$

$$\hat{g}_{\alpha+1}(t,s) = \int_s^t \hat{g}_\alpha(\rho(\tau),s) \nabla \tau, \quad \forall s,t \in T. \quad (1.7)$$

We are motivated by the formula

$$\int_t^x \frac{(x-s)^{\mu-1}(s-t)^{\nu-1}}{\Gamma(\mu)\Gamma(\nu)} ds = \frac{(x-t)^{\mu+\nu-1}}{\Gamma(\mu+\nu)}, \quad (1.8)$$

where $\mu, \nu > 0$ and $\Gamma$ the gamma function.
Assumption 4 Let \( \alpha, \beta > 1 \) and \( x < t \leq \tau, x, t, \tau \in \mathbb{T} \). We assume that

\[
\int_x^\rho (\tau) g_{\alpha-1} (\rho (t), x) g_{\beta-1} (\rho (\tau), t) \nabla t = g_{\alpha+\beta-1} (\rho (\tau), x).
\]  

We call for \( \alpha, \beta > 1 \) and \( x < t \leq \tau \),

\[
\gamma (x, \tau) := \int_\rho (\tau) g_{\alpha-1} (\rho (t), x) g_{\beta-1} (\rho (\tau), t) \nabla t.
\]

It holds

\[
\gamma (x, \tau) = \nu (\tau) g_{\alpha-1} (\rho (\tau), x) g_{\beta-1} (\rho (\tau), \tau),
\]

where \( \nu (\tau) := \tau - \rho (\tau) \), the backward graininess, see [9], p. 332, under the assumption \( \mathbb{T} = \mathbb{T}_k \).

2 Results

We need

Definition 5 Let \( a, b \in \mathbb{T}, \alpha \geq 1 \) and \( f : [a, b] \cap \mathbb{T} \to \mathbb{R} \). Here \( f \in L_1 ((a, b] \cap \mathbb{T}) \) (Lebesgue \( \nabla \)-integrable function on \( (a, b] \cap \mathbb{T} \)). We define the right \( \nabla \)-Riemann-Liouville type fractional integral

\[
J^1_{\alpha-} f (t) := \int_t^b g_{\alpha-1} (\rho (\tau), t) f (\tau) \nabla \tau,
\]

for \( t \in [a, b] \cap \mathbb{T} \). Here \( J^1_{\alpha-} f (t) \nabla \tau = \int_t^b \nabla \tau \).

By [8] we get that \( J^1_{\alpha-} f (t) \) is absolutely continuous in \( t \in [a, b] \cap \mathbb{T} \).

Lemma 6 Let \( \alpha > 1, f \in L_1 ((a, b] \cap \mathbb{T}), f : [a, b] \cap \mathbb{T} \to \mathbb{R} \). Assume that \( g_{\alpha-1} (\rho (\tau), t) \) is Lebesgue \( \nabla \)-measurable on \( ([a, b] \cap \mathbb{T})^2 \); \( a, b \in \mathbb{T} \). Then \( J^1_{\alpha-} f \in L_1 ([a, b] \cap \mathbb{T}) \), that is \( J^1_{\alpha-} f \) is finite a.e.

Proof. By Tietze’s extension theorem of General Topology we easily derive that the continuous function \( g_{\alpha-1} \) on \( ([a, b] \cap \mathbb{T})^2 \) is bounded, since its continuous extension \( G_{\alpha-1} \) on \( [a, b]^2 \) is bounded. Notice that \( ([a, b] \cap \mathbb{T})^2 \) is a closed subset of \( [a, b]^2 \).

So there exists \( M > 0 \) such that \( |g_{\alpha-1} (s, t)| \leq M, \forall (s, t) \in ([a, b] \cap \mathbb{T})^2 \).

Let \( id \) denote the identity map. We see that

\[
(\rho, id) (([a, b] \cap \mathbb{T}) \times ([a, b] \cap \mathbb{T})) \subseteq ([a, b] \cap \mathbb{T})^2.
\]

Therefore \( |g_{\alpha-1} (\rho (t), t)| \leq M, \forall (t, \tau) \in ([a, b] \cap \mathbb{T}) \times ([a, b] \cap \mathbb{T}) \), since \( (\rho (t), t) \in ([a, b] \cap \mathbb{T})^2 \).

Define \( K : \Omega := ([a, b] \cap \mathbb{T})^2 \to \mathbb{R} \), by

\[
K (\tau, t) := \begin{cases} g_{\alpha-1} (\rho (\tau), t), & \text{if } a \leq t < \tau \leq b, \\
0, & \text{if } a \leq \tau \leq t \leq b, 
\end{cases}
\]
where \( t, \tau \in \mathbb{T} \).

Clearly \( K \) is Lebesgue \( \nabla \)-measurable on \( \Omega \), since the restriction of a measurable function to a measurable subset of its domain is measurable function and the union of two measurable functions over disjoint domains is measurable. Notice that \( |K(\tau,t)| \leq M, \forall (\tau,t) \in ([a,b] \cap \mathbb{T})^2 \).

Next we consider the repeated double Lebesgue \( \nabla \)-integral

\[
\left( \int_a^b \left( \int_a^b |K(\tau,t)| |f(\tau)| \nabla t \right) \nabla \tau = \int_a^b |f(\tau)| \left( \int_a^b |K(\tau,t)| \nabla t \right) \nabla \tau \right.
\]

\[
\leq M(b-a) \int_a^b |f(\tau)| \nabla \tau = M(b-a) \|f\|_{L^1([a,b] \cap \mathbb{T})} < \infty.
\]

By Tonelli’s theorem we derive that \((\tau,t) \rightarrow K(\tau,t) f(\tau)\) is Lebesgue \( \nabla \)-integrable over \( \Omega \).

Let now the characteristic function

\[
\chi_{(t,b]}(\tau) = \begin{cases} 1, & \text{if } \tau \in (t,b] \\ 0, & \text{else,} \end{cases}
\]

where \( \tau \in [a,b] \cap \mathbb{T} \).

Then the function \((\tau,t) \rightarrow \chi_{(t,b]}(\tau) K(\tau,t) f(\tau)\) is Lebesgue \( \nabla \)-integrable over \( \Omega \).

Hence by Fubini’s theorem we get that

\[
\int_a^b \chi_{(t,b]}(\tau) K(\tau,t) f(\tau) \nabla \tau = \int_t^b g_{\alpha-1}(\rho(\tau),t) f(\tau) \nabla \tau = J^\alpha_b f(t)
\]

is Lebesgue \( \nabla \)-integrable on \([a,b] \cap \mathbb{T}\), proving the claim. \( \square \)

We make

**Assumption 7** From now on we assume that \( g_{\alpha-1}(\rho(\cdot),\cdot) \) is continuous on \(([a,b] \cap \mathbb{T})^2\), for any \( \alpha > 1 \).

We give

**Definition 8** Let \( f \in L^1((a,b] \cap \mathbb{T}) \). We define the right backward graininess deviation functional of \( f \) as follows

\[
\theta(f,\alpha,\beta,b,\mathbb{T},x) := \int_x^b f(\tau) \gamma(x,\tau) \nabla \tau.
\]

(2.2)

It holds

\[
\theta(f,\alpha,\beta,b,\mathbb{T},x) = \int_x^b f(\tau) v(\tau) g_{\alpha-1}(\rho(\tau),x) g_{\beta-1}(\rho(\tau),\tau) \nabla \tau,
\]

(2.3)

under the assumption \( \mathbb{T} = \mathbb{T}_k \).

If \( \mathbb{T} = \mathbb{R} \), then \( \theta(f,\alpha,\beta,b,\mathbb{T},x) = 0 \).

We give the following semigroup property of right \( \nabla \)-Riemann-Liouville type fractional integrals.
Theorem 9 Let the time scale $\mathbb{T}$ such that $a, b \in \mathbb{T}$, $f \in L_1((a, b) \cap \mathbb{T})$; $\alpha, \beta > 1$. Then

$$J_{b^-}^{\alpha} J_{b^-}^{\beta} f (x) = J_{b^-}^{\alpha + \beta} f (x), \forall x \in [a, b] \cap \mathbb{T}. \quad (2.4)$$

Proof. For $\beta > 1$ we have

$$J_{b^-}^{\beta} f (t) = \int_t^b \tilde{g}_{\beta - 1} (\rho (\tau), t) f (\tau) \nabla \tau.$$ 

We observe that

$$J_{b^-}^{\alpha} J_{b^-}^{\beta} f (x) = \int_x^b \tilde{g}_{\alpha - 1} (\rho (t), x) J_{b^-}^{\beta} f (t) \nabla t =$$

$$\int_x^b \tilde{g}_{\alpha - 1} (\rho (t), x) \left( \int_t^b \tilde{g}_{\beta - 1} (\rho (\tau), t) f (\tau) \nabla \tau \right) \nabla t =$$

$$\int_x^b \left( \int_t^b \tilde{g}_{\alpha - 1} (\rho (t), x) \tilde{g}_{\beta - 1} (\rho (\tau), t) f (\tau) \nabla \tau \right) \nabla t =: (*).$$

Clearly here it holds

$$|\tilde{g}_{\alpha - 1} (\rho (t), x)| \leq M_1, \forall t, x \in [a, b] \cap \mathbb{T},$$

and

$$|\tilde{g}_{\beta - 1} (\rho (\tau), t)| \leq M_2, \forall \tau, t \in [a, b] \cap \mathbb{T},$$

where $M_1, M_2 > 0$.

Hence

$$\left| J_{b^-}^{\alpha} J_{b^-}^{\beta} f (x) \right| \leq \int_x^b \left( \int_t^b |\tilde{g}_{\alpha - 1} (\rho (t), x)| |\tilde{g}_{\beta - 1} (\rho (\tau), t)| |f (\tau)| \nabla \tau \right) \nabla t \leq$$

$$M_1 M_2 \left( \int_x^b \left( \int_t^b |f (\tau)| \nabla \tau \right) \nabla t \right) \leq M_1 M_2 \left( \int_x^b \left( \int_t^b |f (\tau)| \nabla \tau \right) \nabla t \right) \leq$$

$$M_1 M_2 (b - a) \|f\|_{L_1((a, b) \cap \mathbb{T})} < \infty.$$ 

Therefore $J_{b^-}^{\alpha} J_{b^-}^{\beta} f (x)$ exists, $\forall x \in [a, b] \cap \mathbb{T}$. Consequently by Fubini’s theorem we have

$$(*): \int_x^b \left( \int_t^\tau \tilde{g}_{\alpha - 1} (\rho (t), x) \tilde{g}_{\beta - 1} (\rho (\tau), t) f (\tau) \nabla \tau \right) \nabla \tau =$$

$$\int_x^b f (\tau) \left( \int_t^\tau \tilde{g}_{\alpha - 1} (\rho (t), x) \tilde{g}_{\beta - 1} (\rho (\tau), t) \nabla \tau \right) \nabla \tau$$

$(x < t \leq \tau)$

$$\overset{(1.9)}{=} \int_x^b f (\tau) \left( \tilde{g}_{\alpha + \beta - 1} (\rho (\tau), x) + \int_{\rho (\tau)}^{\tau} \tilde{g}_{\alpha - 1} (\rho (t), x) \tilde{g}_{\beta - 1} (\rho (\tau), t) \nabla t \right) \nabla \tau$$

$$= \int_x^b \tilde{g}_{\alpha + \beta - 1} (\rho (\tau), x) f (\tau) \nabla \tau + \int_x^b f (\tau) \gamma (x, \tau) \nabla \tau$$

$$= J_{b^-}^{\alpha + \beta} f (x) + \int_x^b f (\tau) \gamma (x, \tau) \nabla \tau.$$
So we have that
\[ J_{b-}^\alpha J_{b-}^\beta f(x) = J_{b-}^{\alpha+\beta} f(x) + \int_x^b f(\tau) \gamma(x, \tau) \nabla \tau \]
proving the claim.

We make

**Remark 10** Let \( \mu > 2 : m - 1 < \mu \leq m \in \mathbb{N} \), i.e. \( m = \lceil \mu \rceil \) (ceiling of number), \( \tilde{\nu} = m - \mu \) (0 \( \leq \tilde{\nu} < 1 \)). Let \( f \in C^m_{ld} ([a, b] \cap \mathbb{T}) \). Clearly here (10) \( f^{\tilde{\nu} m} \) is a Lebesgue \( \nabla \)-integrable function. We define the right nabla fractional derivative on \( \mathbb{T} \) of order \( \mu - 1 \) as follows:
\[
\nabla_{b-}^{\mu-1} f(t) = (-1)^m \left( J_{b-}^{\tilde{\nu}+1} f^{\tilde{\nu} m} \right)(t) = (-1)^m \int_t^b \tilde{g}_x(\rho(\tau) , t) f^{\tilde{\nu} m}(\tau) \nabla \tau , \quad (2.5)
\]
\( \forall t \in [a, b] \cap \mathbb{T} \).

Notice \( \nabla_{b-}^{\mu-1} f \in C ([a, b] \cap \mathbb{T}) \), by a simple argument using the dominated convergence theorem in Lebesgue \( \nabla \)-sense.

If \( \mu = m \), then \( \tilde{\nu} = 0 \), then
\[
\nabla_{b-}^{m-1} f(t) = (-1)^m \int_t^b f^{\tilde{\nu} m}(\tau) \nabla \tau = (-1)^m \left( f^{\tilde{\nu} m-1}(b) - f^{\tilde{\nu} m-1}(t) \right) , \quad (2.6)
\]

More generally, by [8], given that \( f^{\tilde{\nu} m-1} \) is everywhere finite and absolutely continuous on \( [a, b] \cap \mathbb{T} \), then \( f^{\tilde{\nu} m} \) exists \( \nabla \)-a.e. and is Lebesgue \( \nabla \)-integrable on \((t, b] \cap \mathbb{T}, \forall t \in [a, b] \cap \mathbb{T} \), and one can plug it into (2.5).

**Remark 11** We observe that
\[
J_{b-}^{\mu-1} \nabla_{b-}^{\mu-1} f(t) = (-1)^m \left( J_{b-}^{\mu-1} J_{b-}^{\tilde{\nu}+1} f^{\tilde{\nu} m} \right)(t) \]
\[= (-1)^m \left( J_{b-}^{\mu-1} J_{b-}^{\tilde{\nu}+1} f^{\tilde{\nu} m} \right)(t) + \theta \left( f^{\tilde{\nu} m}, \mu - 1, \tilde{\nu} + 1, b, \mathbb{T}, t \right)\]
\[= (-1)^m \left( J_{b-}^m f^{\tilde{\nu} m}(t) + \theta \left( f^{\tilde{\nu} m}, \mu - 1, \tilde{\nu} + 1, b, \mathbb{T}, t \right) \right) . \quad (2.7)
\]
Hence we proved that
\[
J_{b-}^{\mu-1} \nabla_{b-}^{\mu-1} f(t) + (-1)^{m+1} \theta \left( f^{\tilde{\nu} m}, \mu - 1, \tilde{\nu} + 1, b, \mathbb{T}, t \right) = \]
\[ (-1)^m \left( J_{b-}^m f^{\tilde{\nu} m}(t) \right) = (-1)^m \left( \int_t^b \tilde{g}_m(\rho(\tau) , t) f^{\tilde{\nu} m}(\tau) \nabla \tau \right) \]
\[= \left( R_m^b \left( f \right) \right)(t) , \quad (1.6)
\]
under the assumption \( \mathbb{T} = \mathbb{T}_k = \mathbb{T}^k \).

We have established the following right nabla time scales Taylor formula.
Theorem 12 Assume $T = T_k = k^l$. Let $f \in C_{id}^m (T)$, $m \in \mathbb{N}$, $a, b \in T$, with $\mu > 2$, $m - 1 < \mu \leq m$, $\tilde{\nu} = m - \mu$. Then

$$f(t) = \sum_{k=0}^{m-1} \hat{h}_k(t,b) f^{\nabla^k}(b) + J_0^{\nabla -1} f(t) + (-1)^{m+1} \theta \left( f^{\nabla m}, \mu - 1, \tilde{\nu} + 1, b, T, t \right),$$

(2.9)

$\forall t \in [a,b] \cap T$.

Remark 13 One can rewrite (2.9) as follows

$$f(t) = \sum_{k=0}^{m-1} \hat{h}_k(t,b) f^{\nabla^k}(b) +$$

$$(-1)^{m+1} \int_t^b f^{\nabla m}(\tau) \nu(\tau) \tilde{g}_{\mu-2}(\rho(\tau), t) \tilde{g}_{\tilde{\nu}}(\rho(\tau), \tau) \nabla \tau$$

$$+ \int_t^b \tilde{g}_{\mu-2}(\rho(\tau), t) \left( \nabla_0^{\mu-1} f \right)(\tau) \nabla \tau,$$

(2.10)

$\forall t \in [a,b] \cap T$.

Corollary 14 In the assumptions of Theorem 12, additionally assume that $f^{\nabla^k}(b) = 0$, $k = 0, 1, \ldots, m - 1$. Then

$$A(t) := f(t) + (-1)^{m} \theta \left( f^{\nabla m}, \mu - 1, \tilde{\nu} + 1, b, T, t \right)$$

$$= \int_t^b \tilde{g}_{\mu-2}(\rho(\tau), t) \left( \nabla_0^{\mu-1} f \right)(\tau) \nabla \tau,$$

(2.11)

$\forall t \in [a,b] \cap T$.

Remark 15 Notice (by [8]) that $\left( J_0^{\nabla -1} \nabla_0^{\mu - 1} f \right)(t)$ and $\theta(f^{\nabla m}, \mu - 1, \tilde{\nu} + 1, b, T, t)$ are absolutely continuous functions on $[a,b] \cap T$.

One can use (2.10) and (2.11) to establish right fractional nabla inequalities on time scales of Poincaré type, Sobolev type, Opial type, Ostrowski type and Hilbert-Pachpatte type, etc, analogous to [1]. To keep the article short we avoid this similar task.

Our theory is not void because it is valid when $T = \mathbb{R}$, see also [1].

References


