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ALMOST PERIODIC SOLUTIONS FOR HYPERBOLIC SEMILINEAR EVOLUTION EQUATIONS

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Abstract. In this paper we study the existence of almost periodic solutions for the semilinear evolution equation

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = Au(t) + f(t, u(t)), \quad t \in \mathbb{R},$$

under the sectoriality of A, a linear operator with not necessarily dense domain, in a Banach space $\mathbb{X}$ and $\sigma(A) \cap i\mathbb{R} = \emptyset$. We use the contraction mapping principle to show the existence and uniqueness of an almost periodic solution in an intermediate space $\mathbb{X}_\alpha$, when the function $f : \mathbb{R} \times \mathbb{X}_\alpha \mapsto \mathbb{X}$ is Stepanov-almost periodic.

Keywords: Sectorial operator; analytic semigroup; hyperbolic semigroup; Stepanov-almost periodic; almost periodic.

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1 Introduction

Let $(\mathbb{X}, \|\cdot\|)$ be a complex Banach space and let $\mathbb{X}_\alpha, \alpha \in (0, 1)$, be an abstract intermediate Banach space between $D(A)$, the domain of a linear operator A defined on $\mathbb{X}$, and $\mathbb{X}$. Examples of $\mathbb{X}_\alpha$ are

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$D((-A^{\alpha}))$, the domains of the fractional powers of $-A$, the real interpolation spaces $D_A(\alpha,\infty)$, the abstract Hölder spaces $D_A(\alpha)$, see A. Lunardi [7] for details.

In this paper, we study the existence and uniqueness of an almost periodic solution to the semilinear evolution equation

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = Au(t) + f(t, u(t)) \tag{1.1}$$

for $t \in \mathbb{R}$ and $u \in AP(\mathbb{R};\mathbb{X}_\alpha)$, where $AP(\mathbb{R};\mathbb{X}_\alpha)$ be the set of all almost periodic functions from $\mathbb{R}$ to $\mathbb{X}_\alpha$, A is an unbounded sectorial operator with not necessarily dense domain in a Banach space $\mathbb{X}$ and $f : \mathbb{R}\times\mathbb{X}_\alpha \mapsto \mathbb{X}$ is a Stepanov-almost periodic function.

The existence of almost periodic solutions of abstract differential equations has been considered by many authors; see [1, 8, 10, 11, 12]. Zaidman [11] considered the equation (1.1) in a Banach space $\mathbb{X}$ and proved the existence and uniqueness of an almost periodic solution, when A is an infinitesimal generator of a C_0-semigroup and $f : \mathbb{R}\times\mathbb{X} \mapsto \mathbb{X}$ is almost periodic function. Boulite, Maniar and N'Guérékata [2] considered the same equation (1.1) and proved the existence and uniqueness of an almost automorphic solution in an intermediate space $\mathbb{X}_\alpha$, when the function $f : \mathbb{R}\times\mathbb{X}_\alpha \mapsto \mathbb{X}$ is almost automorphic.

In this paper, we extend the previous-mentioned results to the equation (1.1). We use the contraction mapping principle to prove the existence and uniqueness of an almost periodic solution of the equation (1.1).

2 Preliminaries

In this section we give some basic definitions, notations, and results. In the rest of this paper, $(\mathbb{X},\|\cdot\|)$ stands for a complex Banach space, A is a sectorial linear operator, which is not necessarily densely defined. Now if A is a linear operator on $\mathbb{X}$, then $\rho(A), \sigma(A), D(A), N(A), R(A)$ stand for the resolvent, spectrum, domain, kernel, and range of A. The space $B(\mathbb{X},\mathbb{Y})$ denotes the Banach space of all bounded linear operators from $\mathbb{X}$ into $\mathbb{Y}$ equipped with its natural norm.

Definition 2.1 *A continuous function $f : \mathbb{R} \mapsto \mathbb{X}$ is said to be almost periodic if for every $\epsilon > 0$ there exists a positive number l such that every interval of length l contains a number τ such that*

$$\|f(t+\tau) - f(t)\| < \epsilon \quad \forall t \in \mathbb{R}.$$

Let $AP(\mathbb{R};\mathbb{X})$ be the set of all almost periodic functions from $\mathbb{R}$ to $\mathbb{X}$. Then $(AP(\mathbb{R};\mathbb{X}),\|\cdot\|_\infty)$ is a Banach space with supremum norm given by

$$\|u\|_\infty = \sup_{t\in\mathbb{R}}\|u(t)\|.$$

Let $\mathbb{Y}$ be a complex Banach space. We define the set $AP(\mathbb{R}\times\mathbb{X};\mathbb{Y})$ which consists of all continuous functions $f : \mathbb{R}\times\mathbb{X} \mapsto \mathbb{Y}$ such that $f(\cdot,x) \in AP(\mathbb{R};\mathbb{Y})$ uniformly for each $x \in E$, where E is any compact subset of $\mathbb{X}$.

Let $1 \le p < \infty$, and denote by $L^p_{\mathrm{loc}}(\mathbb{R};\mathbb{X})$ the space of all functions from $\mathbb{R}$ into $\mathbb{X}$ which are locally p-integrable in Bochner-Lebesgue sense. We say that a function, $f \in L^p_{\mathrm{loc}}(\mathbb{R};\mathbb{X})$ is

p-Stepanov bounded (S^p-bounded) if

$$\|f\|_{S^p} = \sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} \|f(s)\|^p \,\mathrm{d}s \Big)^{1/p} < \infty.$$

We indicate by $L_s^p(\mathbb{R};\mathbb{X})$ the set of S^p-bounded functions from $\mathbb{R}$ to $\mathbb{X}$.

Definition 2.2 *A function $f \in L_s^p(\mathbb{R};\mathbb{X})$ is said to be almost periodic in the sense of Stepanov (S^p-almost periodic) if for every $\epsilon > 0$ there exists a positive number l such that every interval of length l contains a number τ such that*

$$\sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} \|f(s+\tau) - f(s)\|^p \,\mathrm{d}s \Big)^{1/p} \le \epsilon.$$

Let $S_{ap}^p(\mathbb{R};\mathbb{X})$ be the set of all S^p-almost periodic functions from $\mathbb{R}$ to $\mathbb{X}$.

It is clear that $f(t)$ almost periodic implies $f(t)$ is S^p-almost periodic; that is, $AP(\mathbb{R};\mathbb{X}) \subset S_{ap}^p(\mathbb{R};\mathbb{X})$. Moreover, if $1 \le m < p$, then $f(t)$ is S^p-almost periodic implies $f(t)$ is S^m-almost periodic.

We define the set $S_{ap}^p(\mathbb{R}\times\mathbb{X};\mathbb{Y})$ which consists of all functions $f : \mathbb{R}\times\mathbb{X} \mapsto \mathbb{Y}$ such that $f(\cdot,x) \in S_{ap}^p(\mathbb{R};\mathbb{Y})$ uniformly for each $x \in E$, where E is any compact subset of $\mathbb{X}$.

Proposition 2.3 *[8, Proposition 3.1] If $f \in S_{ap}^p(\mathbb{R}\times\mathbb{X};\mathbb{Y})$ and $g \in AP(\mathbb{R};\mathbb{X})$, then $f(\cdot,g(\cdot)) \in S_{ap}^p(\mathbb{R};\mathbb{Y})$.*

Definition 2.4 *A linear operator $A : D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ (not necessarily densely defined) is said to be sectorial if there are constants $\omega \in \mathbb{R}, \theta \in (\frac{\pi}{2},\pi)$ and $M > 0$ such that*

$$\rho(A) \supset S_{\theta,\omega} := \{\lambda \in \mathbb{C} : \lambda \ne \omega, |\arg(\lambda-\omega)| < \theta\} \quad \textit{and}$$

$$\|R(\lambda,A)\| \le \frac{M}{|\lambda-\omega|}, \quad \lambda \in S_{\theta,\omega}.$$

It is known that if A is sectorial, then it generates an analytic semigroup $(T(t))_{t\ge 0}$, which maps $(0,\infty)$ into $B(\mathbb{X})$ and such that there exist $M_0, M_1 > 0$ with

$$\|T(t)\| \le M_0 e^{\omega t} \quad \text{for } t > 0, \tag{2.1}$$
$$\|t(A-\omega I)T(t)\| \le M_1 e^{\omega t} \quad \text{for } t > 0 \tag{2.2}$$

where $I : \mathbb{X} \to \mathbb{X}$ is the identity map.

Throughout the rest of the paper, we assume that the semigroup $(T(t))_{t\ge 0}$ is hyperbolic; that is, there exists a projection P and constants $M, \delta > 0$ such that $T(t)$ commutes with P, satisfies $T(t)N(P) = N(P)$, $T(t) : R(Q) \mapsto R(Q)$ is invertible, and the following hold.

$$\|T(t)Px\| \le Me^{-\delta t}\|x\| \quad \text{for} \quad t \ge 0, \tag{2.3}$$
$$\|T(t)Qx\| \le Me^{\delta t}\|x\| \quad \text{for} \quad t \le 0, \tag{2.4}$$

where $Q := I - P$ and, for $t \le 0, T(t) := (T(-t))^{-1}$.

We recall that the analytic semigroup $(T(t))_{t\geq 0}$ associated with A is hyperbolic if and only if

$$\sigma(A) \cap i\mathbb{R} = \emptyset,$$

see for instance [4, Prop. 1.15, p. 305].

Definition 2.5 *Let $\alpha \in (0,1)$. A Banach space $(\mathbb{X}_\alpha, \|\cdot\|_\alpha)$ is said to be an intermediate space between $D(A)$ and $\mathbb{X}$, or a space of class $\mathcal{J}_\alpha$, if $D(A) \subset \mathbb{X}_\alpha \subset \mathbb{X}$ and there is a constant $c > 0$ such that*

$$\|x\|_\alpha \leq c\,\|x\|^{1-\alpha}\,\|x\|_A^\alpha, \quad x \in D(A), \tag{2.5}$$

where $\|\cdot\|_A$ is the graph norm of A.

Examples of $\mathbb{X}_\alpha$ are $D((-A^\alpha))$ for $\alpha \in (0,1)$, the domains of the fractional powers of $-A$, the real interpolation spaces $D_A(\alpha,\infty), \alpha \in (0,1)$, defined as follows

$$\begin{cases} D_A(\alpha,\infty) := \{x \in \mathbb{X} : [x]_\alpha = \sup\limits_{0<t\leq 1} \|t^{1-\alpha} AT(t)x\| < \infty\}, \\ \|x\|_\alpha = \|x\| + [x]_\alpha, \end{cases}$$

and the abstract Hölder spaces $D_A(\alpha) := \overline{D(A)}^{\|\cdot\|_\alpha}$.

For the hyperbolic analytic semigroup $(T(t))_{t\geq 0}$, we can easily check that similar estimations as both (2.3) and (2.4) still hold with norms $\|\cdot\|_\alpha$. In fact, as the part of A in $R(Q)$ is bounded, it follows from (2.4) that

$$\|AT(t)Qx\| \leq C' e^{\delta t}\|x\| \quad \text{for} \ \ t < 0.$$

Hence, from (2.5) there exists a constant $c(\alpha) > 0$ such that

$$\|T(t)Qx\|_\alpha \leq c(\alpha) e^{\delta t}\|x\| \quad \text{for} \quad t \leq 0. \tag{2.6}$$

In addition to the above, the following holds

$$\|T(t)Px\|_\alpha \leq \|T(1)\|_{B(\mathbb{X},\mathbb{X}_\alpha)}\|T(t-1)Px\| \ \text{ for } t \geq 1,$$

and hence from (2.3), one obtains

$$\|T(t)Px\|_\alpha \leq M' e^{-\delta t}\|x\|, \ \text{ for } t \geq 1,$$

where M' depends on α. For $t \in (0,1]$, by (2.2) and (2.5)

$$\|T(t)Px\|_\alpha \leq M'' t^{-\alpha}\|x\|.$$

Hence, there exist constants $M(\alpha) > 0$ and $\gamma > 0$ such that

$$\|T(t)Px\|_\alpha \leq M(\alpha) t^{-\alpha} e^{-\gamma t}\|x\| \quad \text{for} \quad t > 0. \tag{2.7}$$

Throughout the rest of the paper we consider the following assumptions.

(H1) The operator A is sectorial and generates a hyperbolic analytic semigroup $(T(t))_{t\geq 0}$.

(H2) Let $1 < p < \infty$, and $f \in S^p_{ap}(\mathbb{R} \times \mathbb{X}_\alpha; \mathbb{X})$.

(H3) The function f is uniformly Lipschitz with respect to the second argument; that is, there exists $K > 0$ such that

$$\|f(t,x) - f(t,y)\| \le K\|x-y\|_\alpha$$

for all $t \in \mathbb{R}$ and for $x, y \in \mathbb{X}_\alpha$.

Definition 2.6 *By an almost periodic mild solution $u : \mathbb{R} \mapsto \mathbb{X}_\alpha$ of the differential equation* (1.1) *we mean that $u \in AP(\mathbb{R}; \mathbb{X}_\alpha)$, and $u(t)$ satisfies*

$$u(t) = \int_{-\infty}^{t} T(t-s)Pf(s,u(s))\,\mathrm{d}s - \int_{t}^{\infty} T(t-s)Qf(s,u(s))\,\mathrm{d}s, \quad t \in \mathbb{R}. \tag{2.8}$$

3 Main results

In this section we prove the existence and uniqueness of almost periodic mild solution for (1.1). We define the mappings Λ, Λ_1 and Λ_2 by

$$(\Lambda u)(t) = \int_{-\infty}^{t} T(t-s)Pf(s,u(s))\,\mathrm{d}s - \int_{t}^{\infty} T(t-s)Qf(s,u(s))\,\mathrm{d}s, \tag{3.1}$$

$$(\Lambda_1 u)(t) = \int_{-\infty}^{t} T(t-s)Pu(s)\,\mathrm{d}s, \tag{3.2}$$

$$(\Lambda_2 u)(t) = \int_{t}^{\infty} T(t-s)Qu(s)\,\mathrm{d}s, \quad t \in \mathbb{R}. \tag{3.3}$$

Throughout the rest of the paper we indicate the conjugate index of p by q; that is, $\frac{1}{p} + \frac{1}{q} = 1$. We show the following.

Lemma 3.1 *If $h \in S^p_{ap}(\mathbb{R}; \mathbb{X})$, then $\Lambda_1 h \in AP(\mathbb{R}; \mathbb{X}_\alpha)$.*

Proof. We consider

$$(\Lambda_1 h)_k(t) = \int_{t-k}^{t-k+1} T(t-s)Ph(s)\,\mathrm{d}s, \quad k \in \mathbb{N},\ t \in \mathbb{R}.$$

Then

$$\begin{aligned}
\|(\Lambda_1 h)_k(t)\|_\alpha &\le \int_{t-k}^{t-k+1} \|T(t-s)Ph(s)\|_\alpha\,\mathrm{d}s \\
&\le M(\alpha)\int_{t-k}^{t-k+1} (t-s)^{-\alpha} e^{-\gamma(t-s)} \|h(s)\|\,\mathrm{d}s \\
&\le M(\alpha)\Big(\int_{t-k}^{t-k+1} (t-s)^{-q\alpha} e^{-q\gamma(t-s)}\,\mathrm{d}s\Big)^{1/q} \Big(\int_{t-k}^{t-k+1} \|h(s)\|^p\,\mathrm{d}s\Big)^{1/p} \\
&\le M(\alpha)\Big(\int_{t-k}^{t-k+1} (t-s)^{-q\alpha} e^{-q\gamma(t-s)}\,\mathrm{d}s\Big)^{1/q} \|h\|_{S^p}.
\end{aligned}$$

We observe that

$$0 < \int_{t-1}^{t} (t-s)^{-q\alpha} e^{-q\gamma(t-s)}\,\mathrm{d}s = \int_0^1 s^{-q\alpha} e^{-q\gamma s}\,\mathrm{d}s < \infty$$

and

$$\begin{aligned}\Big(\int_{t-k}^{t-k+1} (t-s)^{-q\alpha} e^{-q\gamma(t-s)}\,\mathrm{d}s\Big)^{1/q} &\le \sup_{t-k\le s\le t-k+1} (t-s)^{-\alpha} e^{-\gamma(t-s)}\\ &= (k-1)^{-\alpha} e^{-\gamma(k-1)} \quad \forall k \ge 2.\end{aligned}$$

Since the series $\sum_{k=2}^{\infty} \dfrac{e^{-\gamma(k-1)}}{(k-1)^{\alpha}}$ is convergent, therefore by the comparison test the series $\sum_{k=1}^{\infty} \Big(\int_{t-k}^{t-k+1} (t-s)^{-q\alpha} e^{-q\gamma(t-s)}\,\mathrm{d}s\Big)^{1/q} = \sum_{k=1}^{\infty} \Big(\int_{k-1}^{k} z^{-q\alpha} e^{-q\gamma z}\,\mathrm{d}z\Big)^{1/q}$ is also convergent. Hence from the Weierstrass test the sequence of functions $\sum_{k=1}^{n} (\Lambda_1 h)_k(t)$ is uniformly convergent on $\mathbb{R}$. Thus we have

$$(\Lambda_1 h)(t) = \sum_{k=1}^{\infty} (\Lambda_1 h)_k(t).$$

Let $\epsilon > 0$. Then there exists a positive number l such that every interval of length l contains a number τ such that

$$\sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} \|h(s+\tau) - h(s)\|^p\,\mathrm{d}s\Big)^{1/p} \le \epsilon_1,$$

where $\epsilon_1 > 0$ satisfies

$$\epsilon_1 M(\alpha) \sum_{k=1}^{\infty} \Big(\int_{k-1}^{k} z^{-q\alpha} e^{-q\gamma z}\,\mathrm{d}z\Big)^{1/q} < \epsilon.$$

Now we consider $\|(\Lambda_1 h)_k(s+\tau) - (\Lambda_1 h)_k(s)\|_\alpha$

$$\begin{aligned}&= \Big\| \int_{s+\tau-k}^{s+\tau-k+1} T(s+\tau-z) P h(z)\,\mathrm{d}z - \int_{s-k}^{s-k+1} T(s-z) P h(z)\,\mathrm{d}z \Big\|_\alpha\\ &\le \int_{s-k}^{s-k+1} \|T(s-z) P[h(\tau+z) - h(z)]\|_\alpha\,\mathrm{d}z\\ &\le M(\alpha) \int_{s-k}^{s-k+1} (s-z)^{-\alpha} e^{-\gamma(s-z)} \|h(\tau+z) - h(z)\|\,\mathrm{d}z\\ &\le M(\alpha) \Big(\int_{s-k}^{s-k+1} (s-z)^{-q\alpha} e^{-q\gamma(s-z)}\,\mathrm{d}z\Big)^{1/q} \Big(\int_{s-k}^{s-k+1} \|h(z+\tau) - h(z)\|^p\,\mathrm{d}z\Big)^{1/p}\\ &\le \epsilon_1 M(\alpha) \Big(\int_{s-k}^{s-k+1} (s-z)^{-q\alpha} e^{-q\gamma(s-z)}\,\mathrm{d}z\Big)^{1/q}\\ &= \epsilon_1 M(\alpha) \Big(\int_{k-1}^{k} z^{-q\alpha} e^{-q\gamma z}\,\mathrm{d}z\Big)^{1/q}.\end{aligned}$$

Therefore,

$$\sum_{k=1}^{\infty} \|(\Lambda_1 h)_k(s+\tau) - (\Lambda_1 h)_k(s)\|_\alpha \le \epsilon_1 M(\alpha) \sum_{k=1}^{\infty} \Big(\int_{k-1}^{k} z^{-q\alpha} e^{-q\gamma z}\,\mathrm{d}z\Big)^{1/q} < \epsilon.$$

Thus we get $\Lambda_1 h \in AP(\mathbb{R};\mathbb{X}_\alpha)$. □

Lemma 3.2 *If $h \in S^p_{ap}(\mathbb{R};\mathbb{X})$, then $\Lambda_2 h \in AP(\mathbb{R};\mathbb{X}_\alpha)$.*

Proof. We consider

$$(\Lambda_2 h)_k(t) = \int_{t+k-1}^{t+k} T(t-s)Qh(s)\,\mathrm{d}s, \quad k \in \mathbb{N},\ t \in \mathbb{R}.$$

Then

$$\begin{aligned}
\|(\Lambda_2 h)_k(t)\|_\alpha &\leq \int_{t+k-1}^{t+k} \|T(t-s)Qh(s)\|_\alpha \,\mathrm{d}s \\
&\leq c(\alpha)\int_{t+k-1}^{t+k} e^{\delta(t-s)}\|h(s)\|\,\mathrm{d}s \\
&\leq c(\alpha)\Big(\int_{t+k-1}^{t+k} e^{q\delta(t-s)}\,\mathrm{d}s\Big)^{1/q}\Big(\int_{t+k-1}^{t+k}\|h(s)\|^p\,\mathrm{d}s\Big)^{1/p} \\
&\leq \frac{c(\alpha)}{\sqrt[q]{q\delta}}\Big(e^{q\delta(1-k)} - e^{-q\delta k}\Big)^{1/q}\|h\|_{S^p} \\
&= \frac{c(\alpha)\sqrt[q]{e^{q\delta}-1}}{\sqrt[q]{q\delta}}e^{-\delta k}\|h\|_{S^p}
\end{aligned}$$

Since the series $\sum_{k=1}^{\infty} e^{-\delta k}$ is convergent, therefore from the Weierstrass test the sequence of functions $\sum_{k=1}^{n}(\Lambda_2 h)_k(t)$ is uniformly convergent on $\mathbb{R}$. Hence we have

$$(\Lambda_2 h)(t) = \sum_{k=1}^{\infty}(\Lambda_2 h)_k(t).$$

Let $\epsilon > 0$. Then there exists a positive number l such that every interval of length l contains a number τ such that

$$\sup_{t\in\mathbb{R}}\Big(\int_t^{t+1}\|h(s+\tau)-h(s)\|^p\,\mathrm{d}s\Big)^{1/p} \leq \epsilon_1,$$

where

$$0 < \epsilon_1 < \frac{\epsilon(e^\delta - 1)\sqrt[q]{q\delta}}{c(\alpha)\sqrt[q]{e^{q\delta}-1}}.$$

Now we consider $\|(\Lambda_2 h)_k(s+\tau) - (\Lambda_2 h)_k(s)\|_\alpha$

$$\begin{aligned}
&= \Big\|\int_{s+\tau+k-1}^{s+\tau+k} T(s+\tau-z)Qh(z)\,\mathrm{d}z - \int_{s+k-1}^{s+k} T(s-z)Qh(z)\,\mathrm{d}z\Big\|_\alpha \\
&\leq \int_{s+k-1}^{s+k}\|T(s-z)Q[h(\tau+z)-h(z)]\|_\alpha\,\mathrm{d}z \\
&\leq c(\alpha)\int_{s+k-1}^{s+k} e^{\delta(s-z)}\|h(\tau+z)-h(z)\|\,\mathrm{d}z \\
&\leq c(\alpha)\Big(\int_{s+k-1}^{s+k} e^{q\delta(s-z)}\,\mathrm{d}z\Big)^{1/q}\Big(\int_{s+k-1}^{s+k}\|h(z+\tau)-h(z)\|^p\,\mathrm{d}z\Big)^{1/p}
\end{aligned}$$

$$\leq \epsilon_1 \frac{c(\alpha)\sqrt[q]{e^{q\delta}-1}}{\sqrt[q]{q\delta}} e^{-\delta k}.$$

Therefore

$$\sum_{k=1}^{\infty} \|(\Lambda_2 h)_k(s+\tau) - (\Lambda_2 h)_k(s)\|_\alpha \leq \epsilon_1 \frac{c(\alpha)\sqrt[q]{e^{q\delta}-1}}{\sqrt[q]{q\delta}} \sum_{k=1}^{\infty} e^{-\delta k} = \epsilon_1 \frac{c(\alpha)\sqrt[q]{e^{q\delta}-1}}{\sqrt[q]{q\delta}(e^{\delta}-1)} < \epsilon.$$

Thus we get $\Lambda_2 h \in AP(\mathbb{R};\mathbb{X}_\alpha)$. □

Lemma 3.3 *The operator Λ maps $AP(\mathbb{R};\mathbb{X}_\alpha)$ into itself.*

Proof. Let $u \in AP(\mathbb{R};\mathbb{X}_\alpha)$. From Proposition 2.3, we get $f(\cdot,u(\cdot)) \in S^p_{ap}(\mathbb{R};\mathbb{X})$. Hence from Lemma 3.1 and Lemma 3.2, we get $(\Lambda_1 f)(\cdot,u(\cdot)), (\Lambda_2 f)(\cdot,u(\cdot)) \in AP(\mathbb{R};\mathbb{X}_\alpha)$. Thus $\Lambda u \in AP(\mathbb{R};\mathbb{X}_\alpha)$. □

Theorem 3.4 *Suppose* $\left(\frac{M(\alpha)\Gamma(1-\alpha)}{\gamma^{1-\alpha}} + \frac{c(\alpha)}{\delta}\right)K < 1$. *Then* (1.1) *has unique almost periodic mild solution.*

Proof. Let $u,v \in AP(\mathbb{R};\mathbb{X}_\alpha)$. We observed that

$$\|(\Lambda_1 u)(t) - (\Lambda_1 v)(t)\|_\alpha \leq \int_{-\infty}^{t} \|T(t-s)P[f(s,u(s)) - f(s,v(s))]\|_\alpha \,\mathrm{d}s$$
$$\leq M(\alpha)\int_{-\infty}^{t} (t-s)^{-\alpha} e^{-\gamma(t-s)} \|f(s,u(s)) - f(s,v(s))\| \,\mathrm{d}s$$
$$\leq KM(\alpha)\int_{-\infty}^{t} (t-s)^{-\alpha} e^{-\gamma(t-s)} \|u(s)-v(s)\|_\alpha \,\mathrm{d}s$$
$$\leq \frac{KM(\alpha)\Gamma(1-\alpha)}{\gamma^{1-\alpha}} \|u-v\|_{\infty,\alpha}.$$

Similarly,

$$\|(\Lambda_2 u)(t) - (\Lambda_2 v)(t)\|_\alpha \leq \int_{t}^{\infty} \|T(t-s)Q[f(s,u(s)) - f(s,v(s))]\|_\alpha \,\mathrm{d}s$$
$$\leq c(\alpha)\int_{t}^{\infty} e^{\delta(t-s)} \|f(s,u(s)) - f(s,v(s))\| \,\mathrm{d}s$$
$$\leq Kc(\alpha)\int_{t}^{\infty} e^{\delta(t-s)} \|u(s)-v(s)\|_\alpha \,\mathrm{d}s$$
$$\leq \frac{Kc(\alpha)}{\delta} \|u-v\|_{\infty,\alpha}.$$

Thus

$$\|\Lambda u - \Lambda v\|_{\infty,\alpha} \leq K\Big(\frac{M(\alpha)\Gamma(1-\alpha)}{\gamma^{1-\alpha}} + \frac{c(\alpha)}{\delta}\Big)\|u-v\|_{\infty,\alpha}.$$

Thus Λ is a contraction map on $AP(\mathbb{R};\mathbb{X}_\alpha)$. Therefore, Λ has unique fixed point in $AP(\mathbb{R};\mathbb{X}_\alpha)$; that is, there exist unique $\psi \in AP(\mathbb{R};\mathbb{X}_\alpha)$ such that $\Lambda\psi = \psi$. Therefore equation (1.1) has a unique almost periodic mild solution. □

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