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MULTIDIMENSIONAL OSTROWSKI INEQUALITIES FOR BANACH SPACE VALUED FUNCTIONS

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Abstract. Here we are dealing with smooth functions from a real box to a Banach space. For these we establish vector multivariate sharp Ostrowski type inequalities to all possible directions. In establishing them we prove interesting multivariate vector identities using integration by parts and other basic analytical methods.

Keywords: Multidimensional integral inequality; Ostrowski inequality; sharp inequality; mixed partial derivative; Banach valued functions.

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1 Introduction

In 1938, A. Ostrowski proved the following inequality [11]:

Theorem 1 *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) whose derivative $f' : (a, b) \to \mathbb{R}$ is bounded on (a, b), i.e. $\|f\|'_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - f(x)\right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\,\|f'\|_\infty\,, \tag{1.1}$$

for any $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible.

Since then there has been a lot of activity around these inequalities with important applications to Numerical Analysis and Probability.

This article is also greatly motivated by the following result:

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Theorem 2 *(see [1]). Let* $f \in C^1\left(\prod_{i=1}^k [a_i, b_i]\right)$*, where* $a_i < b_i$*;* $a_i, b_i \in \mathbb{R}$*,* $i = 1, \ldots, k$*, and let* $\overrightarrow{x_0} := (x_{01}, \ldots, x_{0k}) \in \prod_{i=1}^k [a_i, b_i]$ *be fixed. Then*

$$\left| \frac{1}{\prod_{i=1}^k (b_i - a_i)} \int_{a_1}^{b_1} \cdots \int_{a_k}^{b_k} f(z_1, \ldots, z_k)\, \mathrm{d}z_1 \ldots \mathrm{d}z_k - f(\overrightarrow{x_0}) \right|$$

$$\leq \sum_{i=1}^k \frac{(x_{0i} - a_i)^2 + (b_i - x_{0i})^2}{2(b_i - a_i)} \left\| \frac{\partial f}{\partial z_i} \right\|_\infty . \tag{1.2}$$

Inequality (1.2) is sharp, here the optimal function is

$$f^*(z_1, \ldots, z_k) := \sum_{i=1}^k |z_i - x_{0i}|^{\alpha_i}, \ \ \alpha_i > 1.$$

Clearly inequality (1.2) generalizes inequality (1.1) to multidimension.

In this article we establish multivariate Ostrowski inequalities for smooth functions from a real box to a Banach space. These involve the norms $\|\cdot\|_p$, $1 \leq p \leq \infty$. Some of these inequalities are sharp.

2 Background

We follow [13, pp. 83-94]. Let $f(t)$ be a function defined on $[a, b] \subseteq \mathbb{R}$ taking values in a real or complex normed linear space $(X, \|\cdot\|)$. Then $f(t)$ is said to be differentiable at a point $t_0 \in [a, b]$ if the limit

$$f'(t_0) := \lim_{h \to 0} \frac{f(t_0 + h) - f(t_0)}{h} \tag{2.1}$$

exists in X, the convergence is in $\|\cdot\|$. This is called the derivative of $f(t)$ at $t = t_0$.

We call $f(t)$ differentiable on $[a, b]$, iff there exists $f'(t) \in X$ for all $t \in [a, b]$. Similarly and inductively are defined higher order derivatives of f, denoted f'', $f^{(3)}, \ldots, f^{(k)}$, $k \in \mathbb{N}$, just as for numerical functions. For all the properties of derivatives see [13, pp. 83-86].

Let now $(X, \|\cdot\|)$ be a Banach space, and $f : [a, b] \to X$.

We define the vector valued Riemann integral $\int_a^b f(t)\, \mathrm{d}t \in X$ as the limit of the vector valued Riemann sums in X, convergence is in $\|\cdot\|$. The definition is as for the numerical valued functions. If $\int_a^b f(t)\, \mathrm{d}t \in X$ we call f integrable on $[a, b]$. If $f \in C([a, b], X)$, then f is integrable [13, p. 87]. For all the properties of vector valued Riemann integrals see [13, pp. 86-91].

We define the space $C^n([a, b], X)$, $n \in \mathbb{N}$, of n-times continuously differentiable functions from $[a, b]$ into X; here continuity is with respect to $\|\cdot\|$ and defined in the usual way as for numerical functions.

Let $(X, \|\cdot\|)$ be a Banach space and $f \in C^n([a, b], X)$, then we have the vector valued Taylor's formula, see [13, pp. 93-94], and also [12, IV, 9; 47]. It holds

$$f(y) - f(x) - f'(x)(y - x) - \frac{1}{2} f''(x)(y - x)^2 - \cdots - \frac{1}{(n-1)!} f^{(n-1)}(x)(y - x)^{n-1}$$

$$= \frac{1}{(n-1)!} \int_x^y (y - t)^{n-1} f^{(n)}(t)\, \mathrm{d}t, \quad \forall\, x, y \in [a, b] . \tag{2.2}$$

In particular (2.2) is true when $X = \mathbb{R}^m$, $\mathbb{C}^m$, $m \in \mathbb{N}$, etc.

A function $f(t)$ with values in a normed linear space X is said to be piecewise continuous (see [13, p. 85]) on the interval $a \leq t \leq b$ if there exists a partition $a = t_0 < t_1 < t_2 < \cdots < t_n = b$ such that $f(t)$ is continuous on every open interval $t_k < t < t_{k+1}$ and has finite limits $f(t_0 + 0)$, $f(t_1 - 0)$, $f(t_1 + 0)$, $f(t_2 - 0)$, $f(t_2 + 0), \ldots, f(t_n - 0)$.

Here $f(t_k - 0) = \lim_{t \uparrow t_k} f(t)$, $f(t_k + 0) = \lim_{t \downarrow t_k} f(t)$.

The values of $f(t)$ at the points t_k can be arbitrary or even undefined.

A function $f(t)$ with values in normed linear space X is said to be piecewise smooth on $[a, b]$, if it is continuous on $[a, b]$ and has a derivative $f'(t)$ at all but a finite number of points of $[a, b]$, and if $f'(t)$ is piecewise continuous on $[a, b]$ (see [13, p. 85]).

Let $u(t)$ and $v(t)$ be two piecewise smooth functions on $[a, b]$, one a numerical function and the other a vector function with values in Banach space X. Then we have the following integration by parts formula [13, p. 93]

$$\int_a^b u(t)\,\mathrm{d}v(t) = u(t)v(t)|_a^b - \int_a^b v(t)\,\mathrm{d}u(t). \tag{2.3}$$

We mention also the mean value theorem for Banach space valued functions:

Theorem 3 *(see [10, p. 3]). Let $f \in C([a, b], X)$, where X is a Banach space. Assume f' exists on $[a, b]$ and $\|f'(t)\| \leq K$, $a < t < b$, then*

$$\|f(b) - f(a)\| \leq K(b - a). \tag{2.4}$$

Here the multiple Riemann integral of a function from a real box to a Banach space is defined similarly to numerical one, however convergence is with respect to $\|\cdot\|$. Similarly are defined the vector valued partial derivatives as in the numerical case.

We mention the equality of vector valued mixed partial derivatives.

Proposition 4 *(see [9, p. 90, Proposition 4.11]). Let $Q = (a, b) \times (c, d) \subseteq \mathbb{R}^2$ and $f \in C(Q, X)$, where $(X, \|\cdot\|)$ is a Banach space. Assume that $\frac{\partial}{\partial t} f(s, t)$, $\frac{\partial}{\partial s} f(s, t)$ and $\frac{\partial^2}{\partial t \partial s} f(s, t)$ exist and are continuous for $(s, t) \in Q$, then $\frac{\partial^2}{\partial s \partial t} f(s, t)$ exists for $(s, t) \in Q$ and*

$$\frac{\partial^2}{\partial s\,\partial t} f(s, t) = \frac{\partial^2}{\partial t\,\partial s} f(s, t), \text{ for } (s, t) \in Q.$$

Notice also that

$$\left(\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b - a)^2}\right)(b - a) = \frac{(x - a)^2 + (b - x)^2}{2\,(b - a)}, \ \ \forall\, x \in [a, b]. \tag{2.5}$$

3 Main Results

Here we present the first vector multivariate Ostrowski type inequality, see also the real analog, Theorem 23.1 of [2, p. 507] and [1].

Theorem 5 *Let $(X, \|\cdot\|)$ be a Banach space and $f \in C^1\left(\prod_{i=1}^k [a_i, b_i], X\right)$, $a_i < b_i$; $a_i, b_i \in \mathbb{R}$, $i = 1, \ldots, k$, and let $\overrightarrow{x_0} := (x_{01}, \ldots, x_{0k}) \in \prod_{i=1}^k [a_i, b_i]$ be fixed. Denote*

$$\left\|\left\|\frac{\partial f}{\partial z_i}\right\|\right\|_\infty := \sup_{x \in \prod_{i=1}^k [a_i, b_i]} \left\|\frac{\partial f(x)}{\partial z_i}\right\|.$$

Then

$$\left\|\frac{1}{\prod_{i=1}^{k}(b_i-a_i)}\int_{a_1}^{b_1}\dots\int_{a_k}^{b_k} f(z_1,\dots,z_k)\,\mathrm{d}z_1\dots\mathrm{d}z_k - f(\overrightarrow{x_0})\right\| \leq \sum_{i=1}^{k}\frac{(x_{0i}-a_i)^2+(b_i-x_{0i})^2}{2(b_i-a_i)}\left\|\left\|\frac{\partial f}{\partial z_i}\right\|\right\|_\infty. \tag{3.1}$$

Inequality (3.1) is sharp, the optimal function is

$$f^*(z_1,\dots,z_k) := \left(\sum_{i=1}^{k}|z_i-x_{0i}|^{\alpha_i}\right)\cdot i_0, \quad \alpha_i>1,$$

where i_0 is a fixed unit vector in X.

Proof. Set $\overrightarrow{z} := (z_1,\dots,z_k)$. Consider $g_{\overrightarrow{z}}(t) := f(\overrightarrow{x_0}+t(\overrightarrow{z}-\overrightarrow{x_0}))$, $t\geq 0$. Note that $g_{\overrightarrow{z}}(0)=f(\overrightarrow{x_0})$, $g_{\overrightarrow{z}}(1)=f(\overrightarrow{z})$. Hence

$$\|f(\overrightarrow{z})-f(\overrightarrow{x_0})\| = \|g_{\overrightarrow{z}}(1)-g_{\overrightarrow{z}}(0)\| \leq \left\|\left\|g'_{\overrightarrow{z}}(\xi)\right\|\right\|_{\infty,(0,1)}(1-0) = \left\|\left\|g'_{\overrightarrow{z}}(\xi)\right\|\right\|_{\infty,(0,1)}.$$

Since

$$g'_{\overrightarrow{z}}(\xi) = (z_1-x_{01})\frac{\partial f}{\partial z_1}(x_0+\xi(z-x_0))+\dots+(z_k-x_{0k})\frac{\partial f}{\partial z_k}(x_0+\xi(z-x_0))$$

we get

$$\|f(\overrightarrow{z})-f(\overrightarrow{x_0})\| \leq \sum_{i=1}^{k}|z_i-x_{0i}|\left\|\left\|\frac{\partial f}{\partial z_i}(\overrightarrow{x_0}+\xi(\overrightarrow{z}-\overrightarrow{x_0}))\right\|\right\|_{\infty,(0,1)} \leq \sum_{i=1}^{k}|z_i-x_{0i}|\left\|\left\|\frac{\partial f}{\partial z_i}\right\|\right\|_\infty. \tag{3.2}$$

Next we see that

$$\begin{aligned}
&\left\|\frac{1}{\prod_{i=1}^{k}(b_i-a_i)}\int_{a_1}^{b_1}\dots\int_{a_k}^{b_k} f(z_1,\dots,z_k)\,\mathrm{d}z_1\dots\,\mathrm{d}z_k - f(\overrightarrow{x_0})\right\| \\
&= \frac{1}{\prod_{i=1}^{k}(b_i-a_i)}\left\|\int_{a_1}^{b_1}\dots\int_{a_k}^{b_k}(f(\overrightarrow{z})-f(\overrightarrow{x_0}))\ \mathrm{d}\overrightarrow{z}\right\| \\
&\leq \frac{1}{\prod_{i=1}^{k}(b_i-a_i)}\int_{a_1}^{b_1}\dots\int_{a_k}^{b_k}\|f(\overrightarrow{z})-f(\overrightarrow{x_0})\|\ \mathrm{d}\overrightarrow{z} \\
&\overset{(3.2)}{\leq} \frac{1}{\prod_{i=1}^{k}(b_i-a_i)}\int_{a_1}^{b_1}\dots\int_{a_k}^{b_k}\left(\sum_{i=1}^{k}|z_i-x_{0i}|\left\|\left\|\frac{\partial f}{\partial z_i}\right\|\right\|_\infty\right)\mathrm{d}z_1\dots\,\mathrm{d}z_k \\
&= \frac{1}{\prod_{i=1}^{k}(b_i-a_i)}\left[\sum_{i=1}^{k}\left(\int_{a_1}^{b_1}\dots\int_{a_k}^{b_k}|z_i-x_{0i}|\ \mathrm{d}z_1\dots\,\mathrm{d}z_k\right)\left\|\left\|\frac{\partial f}{\partial z_i}\right\|\right\|_\infty\right] =: (*).
\end{aligned}$$

Here notice that

$$\int_{a_i}^{b_i}|z_i-x_{0i}|\ \mathrm{d}z_i = \frac{(x_{0i}-a_i)^2+(b_i-x_{0i})^2}{2},\ i=1,\dots,k. \tag{3.3}$$

Therefore, by (3.3) we obtain

$$(*) = \frac{1}{\prod_{j=1}^{k}(b_j - a_j)} \cdot \sum_{i=1}^{k} \left\|\left\|\frac{\partial f}{\partial z_i}\right\|\right\|_\infty \frac{(x_{0i} - a_i)^2 + (b_i - x_{0i})^2}{2} \prod_{\substack{j=1 \\ j\neq i}}^{k}(b_j - a_j)$$
$$= \sum_{i=1}^{k} \frac{(x_{0i} - a_i)^2 + (b_i - x_{0i})^2}{2\,(b_i - a_i)} \left\|\left\|\frac{\partial f}{\partial z_i}\right\|\right\|_\infty,$$

so that we establish inequality (3.1).

In the following we prove the sharpness of (3.1): Notice that $f^*(\overrightarrow{x_0}) = 0$ and

$$\frac{\partial f^*}{\partial z_i}(\overrightarrow{z}) = \alpha_i \left|z_i - x_0\right|^{\alpha_i - 1} \cdot sgn\,(z_i - x_{0i}) \cdot i_0, \;\; \alpha_i > 1.$$

In particular we find

$$\left\|\frac{\partial f^*(\overrightarrow{x})}{\partial z_i}\right\| = \alpha_i \left|z_i - x_{0i}\right|^{\alpha_i - 1},$$

and $(a_i \leq z_i \leq b_i)$

$$\left\|\left\|\frac{\partial f^*}{\partial z_i}\right\|\right\|_\infty = \alpha_i \left(\max\left(b_i - x_{0i}, x_{0i} - a_i\right)\right)^{\alpha_i - 1}.$$

Consequently, we observe

$$R.H.S.(3.1) = \sum_{i=1}^{k} \frac{(x_{0i} - a_i)^2 + (b_i - x_{0i})^2}{2\,(b_i - a_i)} \left\|\left\|\frac{\partial f^*}{\partial z_i}\right\|\right\|_\infty$$
$$= \sum_{i=1}^{k} \frac{(x_{0i} - a_i)^2 + (b_i - x_{0i})^2}{2\,(b_i - a_i)} \alpha_i \left(\max\left(b_i - x_{0i}, x_{0i} - a_i\right)\right)^{\alpha_i - 1},$$

and

$$\lim_{\substack{\alpha_i \to 1 \\ i=1,\ldots,k}} R.H.S.(3.1) = \sum_{i=1}^{k} \frac{(x_{0i} - a_i)^2 + (b_i - x_{0i})^2}{2\,(b_i - a_i)}. \tag{3.4}$$

Moreover, we get that

$$L.H.S.(3.1) = \frac{1}{\prod_{i=1}^{k}(b_i - a_i)} \left(\int_{a_1}^{b_1} \cdots \int_{a_k}^{b_k} \left(\sum_{i=1}^{k} \left|z_i - x_{0i}\right|^{\alpha_i}\right) \mathrm{d}z_1 \ldots \mathrm{d}z_k\right) \|i_0\|$$
$$= \frac{1}{\prod_{i=1}^{k}(b_i - a_i)} \sum_{i=1}^{k} \int_{a_1}^{b_1} \cdots \int_{a_k}^{b_k} \left|z_i - x_{0i}\right|^{\alpha_i} \mathrm{d}z_1 \ldots \mathrm{d}z_k$$
$$= \frac{1}{\prod_{i=1}^{k}(b_i - a_i)} \sum_{i=1}^{k} \frac{(x_{0i} - a_i)^{\alpha_i + 1} + (b_i - x_{0i})^{\alpha_i + 1}}{\alpha_i + 1} \prod_{\substack{j=1 \\ j\neq i}}^{k}(b_j - a_j)$$
$$= \sum_{i=1}^{k} \frac{(x_{0i} - a_i)^{\alpha_i + 1} + (b_i - x_{0i})^{\alpha_i + 1}}{(\alpha_i + 1)\,(b_i - a_i)},$$

and

$$\lim_{\substack{\alpha_i \to 1 \\ i=1,\ldots,k}} L.H.S.(3.1) = \sum_{i=1}^{k} \frac{(x_{0i} - a_i)^2 + (b_i - x_{0i})^2}{2\,(b_i - a_i)}. \tag{3.5}$$

At the end from (3.4) and (3.5) we obtain that

$$\lim_{\substack{\alpha_i \to 1 \\ i=1,\ldots,k}} L.H.S.(3.1) = \lim_{\substack{\alpha_i \to 1 \\ i=1,\ldots,k}} R.H.S.(3.1),$$

proving the inequality (3.1) is sharp. □

Regarding vector higher order derivatives we give the following results:

Theorem 6 *Let $(X, \|\cdot\|)$ be a Banach space and $f \in C^{n+1}\left(\prod_{i=1}^{k} [a_i, b_i], X\right)$, $n \in \mathbb{N}$ and fixed $\overrightarrow{x_0} \in \prod_{i=1}^{k} [a_i, b_i]$, $k \geq 1$, such that all vector partial derivatives $f_\alpha := \frac{\partial^\alpha f}{\partial z^\alpha}$, where $\alpha = (\alpha_1, \ldots, \alpha_k)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, k$, $|\alpha| = \sum_{i=1}^{k} \alpha_i = j$, $j = 1, \ldots, n$ fulfill $f_\alpha(\overrightarrow{x_0}) = 0$. Then*

$$\left\| \frac{1}{\prod_{i=1}^{k} (b_i - a_i)} \int_{\prod_{i=1}^{k}[a_i,b_i]} f(\overrightarrow{z})\, \mathrm{d}\overrightarrow{z} - f(\overrightarrow{x_0}) \right\|$$
$$\leq \left(\frac{D_{n+1}(f)}{(n+1)! \prod_{i=1}^{k} (b_i - a_i)} \right) \int_{\prod_{i=1}^{k}[a_i,b_i]} \left(\|\overrightarrow{z} - \overrightarrow{x_0}\|_{\ell_1} \right)^{n+1} \mathrm{d}\overrightarrow{z}, \tag{3.6}$$

where

$$D_{n+1}(f) := \max_{\alpha : |\alpha| = n+1} \| \|f\| \|_\infty, \tag{3.7}$$

and

$$\|\overrightarrow{z} - \overrightarrow{x_0}\|_{\ell_1} := \sum_{i=1}^{k} |z_i - x_{0i}|. \tag{3.8}$$

Proof. Take $g_{\overrightarrow{z}}(t) := f(\overrightarrow{x_0} + t(\overrightarrow{z} - \overrightarrow{x_0}))$, $0 \leq t \leq 1$. Notice that $g_{\overrightarrow{z}}(0) = f(\overrightarrow{x_0})$ and $g_{\overrightarrow{z}}(1) = f(\overrightarrow{z})$. The jth derivative of $g_{\overrightarrow{z}}(t)$, based on Proposition 4, is given by

$$g_{\overrightarrow{z}}^{(j)}(t) = \left[\left(\sum_{i=1}^{k} (z_i - x_{0i}) \frac{\partial}{\partial z_i} \right)^j f \right] (x_{01} + t(z_1 - x_{01}), \ldots, x_{0k} + t(z_k - x_{0k})) \tag{3.9}$$

and

$$g_{\overrightarrow{z}}^{(j)}(0) = \left[\left(\sum_{i=1}^{k} (z_i - x_{0i}) \frac{\partial}{\partial z_i} \right)^j f \right] (\overrightarrow{x_0}), \tag{3.10}$$

for $j = 1, \ldots, n+1$.

Let f_α be a partial derivative of $f \in C^{n+1}\left(\prod_{i=1}^{k} [a_i, b_i]\right)$. Because by assumption of the theorem we have $f_\alpha(\overrightarrow{x_0}) = 0$ for all $\alpha : |\alpha| = j$, $j = 1, \ldots, n$, we find that

$$g_{\overrightarrow{z}}^{(j)}(0) = 0, \quad j = 1, \ldots, n.$$

Hence by vector Taylor's theorem (2.2) we see that

$$f(\overrightarrow{z}) - f(\overrightarrow{x_0}) = \sum_{j=1}^{n} \frac{g_{\overrightarrow{z}}^{(j)}(0)}{j!} + R_n(\overrightarrow{z}, 0) = R_n(\overrightarrow{z}, 0), \tag{3.11}$$

where

$$R_n(\overrightarrow{z},0) := \int_0^1 \left(\int_0^{t_1} \cdots \left(\int_0^{t_{n-1}} \left(g_{\overrightarrow{z}}^{(n)}(t_n) - g_{\overrightarrow{z}}^{(n)}(0) \right) \mathrm{d}t_n \right) \ldots \mathrm{d}t_2 \right) \mathrm{d}t_1. \tag{3.12}$$

Therefore,

$$\|R_n(\overrightarrow{z},0)\| \leq \int_0^1 \left(\int_0^{t_1} \cdots \left(\int_0^{t_{n-1}} \left\| \left| g_{\overrightarrow{z}}^{(n+1)} \left(\xi(t_n) \right) \right\| \right|_\infty t_n \, \mathrm{d}t_n \right) \ldots \mathrm{d}t_2 \right) \mathrm{d}t_1, \tag{3.13}$$

by the vector mean value Theorem 3 applied on $g_{\overrightarrow{z}}^{(n)}$ over $(0, t_n)$. Moreover, we get

$$\|R_n(\overrightarrow{z},0)\| \leq \left\| \left| g_{\overrightarrow{z}}^{(n+1)} \right\| \right|_{\infty,[0,1]} \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{n-1}} t_n \, \mathrm{d}t_n \ldots \mathrm{d}t_2 \, \mathrm{d}t_1 = \frac{\left\| \left| g_{\overrightarrow{z}}^{(n+1)} \right\| \right|_{\infty,[0,1]}}{(n+1)!}. \tag{3.14}$$

However, there exists a $t_0 \in [0,1]$ such that $\left\| \left| g_{\overrightarrow{z}}^{(n+1)} \right\| \right|_{\infty,[0,1]} = \left\| g_{\overrightarrow{z}}^{(n+1)}(t_0) \right\|$. That is,

$$\begin{aligned} \left\| \left| g_{\overrightarrow{z}}^{(n+1)} \right\| \right|_{\infty,[0,1]} &= \left\| \left[\left(\sum_{i=1}^k (z_i - x_{0i}) \frac{\partial}{\partial z_i} \right)^{n+1} f \right] \left(\overrightarrow{x_0} + t_0 \left(\overrightarrow{z} - \overrightarrow{x_0} \right) \right) \right\| \\ &\leq \left[\left(\sum_{i=1}^k |z_i - x_{0i}| \left\| \frac{\partial}{\partial z_i} \right\| \right)^{n+1} f \right] \left(\overrightarrow{x_0} + t_0 \left(\overrightarrow{z} - \overrightarrow{x_0} \right) \right). \end{aligned}$$

I.e.,

$$\left\| \left| g_{\overrightarrow{z}}^{(n+1)} \right\| \right|_{\infty,[0,1]} \leq \left[\left(\sum_{i=1}^k |z_i - x_{0i}| \left\| \left| \frac{\partial}{\partial z_i} \right\| \right|_\infty \right)^{n+1} f \right]. \tag{3.15}$$

Hence by (3.15) we get

$$\|f(\overrightarrow{z}) - f(\overrightarrow{x_0})\| = \|R_n(\overrightarrow{z},0)\| \leq \frac{1}{(n+1)!} \left[\left(\sum_{i=1}^k |z_i - x_{0i}| \left\| \left| \frac{\partial}{\partial z_i} \right\| \right|_\infty \right)^{n+1} f \right]. \tag{3.16}$$

In the following we observe

$$\begin{aligned} &\left\| \frac{1}{\prod_{i=1}^k (b_i - a_i)} \int_{\prod_{i=1}^k [a_i,b_i]} f(\overrightarrow{z}) \, \mathrm{d}\overrightarrow{z} - f(\overrightarrow{x_0}) \right\| \\ &\quad = \frac{1}{\prod_{i=1}^k (b_i - a_i)} \left\| \int_{\prod_{i=1}^k [a_i,b_i]} \left(f(\overrightarrow{z}) - f(\overrightarrow{x_0}) \right) \mathrm{d}\overrightarrow{z} \right\| \\ &\quad \leq \frac{1}{\prod_{i=1}^k (b_i - a_i)} \int_{\prod_{i=1}^k [a_i,b_i]} \|f(\overrightarrow{z}) - f(\overrightarrow{x_0})\| \, \mathrm{d}\overrightarrow{z} \end{aligned}$$

and by (3.16),

$$\begin{aligned} &\leq \frac{1}{(n+1)! \prod_{i=1}^k (b_i - a_i)} \int_{\prod_{i=1}^k [a_i,b_i]} \left(\left(\sum_{i=1}^k |z_i - x_{0i}| \left\| \left| \frac{\partial}{\partial z_i} \right\| \right|_\infty \right)^{n+1} f \right) \mathrm{d}\overrightarrow{z} \\ &\leq \left(\frac{D_{n+1}(f)}{(n+1)! \prod_{i=1}^k (b_i - a_i)} \right) \int_{\prod_{i=1}^k [a_i,b_i]} \left(\|\overrightarrow{z} - \overrightarrow{x_0}\|_{\ell_1} \right)^{n+1} \mathrm{d}\overrightarrow{z}. \end{aligned} \tag{3.17}$$

This establishes inequality (3.6). □

Corollary 7 *(to Theorem 6) Under the assumptions of Theorem 6 we find that*

$$\left\| \frac{1}{\prod_{i=1}^{k}(b_i - a_i)} \int_{\prod_{i=1}^{k}[a_i,b_i]} f(\overrightarrow{z})\, \mathrm{d}\overrightarrow{z} - f(\overrightarrow{x_0}) \right\|$$
$$\leq \frac{1}{(n+1)!\prod_{i=1}^{k}(b_i - a_i)} \int_{\prod_{i=1}^{k}[a_i,b_i]} \left[\left(\sum_{i=1}^{k} |z_i - x_{0i}| \left\| \left\| \frac{\partial}{\partial z_i} \right\| \right\|_{\infty} \right)^{n+1} f \right] \mathrm{d}\overrightarrow{z}. \tag{3.18}$$

Furthermore, (3.18) is sharp: when n is odd it is attained by

$$f^*(z_1, \ldots, z_k) := \left(\sum_{i=1}^{k} (z_i - x_{0i})^{n+1} \right) \cdot i_0 \tag{3.19}$$

while when n is even the optimal function is

$$\overline{f}(z_1, \ldots, z_k) := \left(\sum_{i=1}^{k} |z_i - x_{0i}|^{n+\alpha_i} \right) \cdot i_0, \ \ \alpha_i > 1, \tag{3.20}$$

where $i_0 \in X : \|i_0\| = 1$.

Proof. Inequality (3.18) comes directly from (3.17). Next we prove the sharpness of (3.18).

i) When n is odd: Notice that $f^*(\overrightarrow{x_0}) = 0$ and

$$\left\| \left\| \frac{\partial^{n+1} f^*}{\partial z_i^{n+1}} \right\| \right\|_{\infty} = (n+1)!,$$

furthermore any mixed partial of f^* equals zero. Thus by plugging f^* into (3.18) we observe that

$$\begin{aligned} R.H.S.(3.18) &= \frac{1}{(n+1)!\ \prod_{i=1}^{k}(b_i - a_i)} \int_{\prod_{i=1}^{k}[a_i,b_i]} \left[\sum_{i=1}^{k} |z_i - x_{0i}|^{n+1} (n+1)! \right] \mathrm{d}\overrightarrow{z} \\ &= \frac{1}{\prod_{i=1}^{k}(b_i - a_i)} \int_{\prod_{i=1}^{k}[a_i,b_i]} \left(\sum_{i=1}^{k} (z_i - x_{0i})^{n+1} \right) \mathrm{d}\overrightarrow{z} \\ &= \frac{1}{\prod_{i=1}^{k}(b_i - a_i)} \int_{\prod_{i=1}^{k}[a_i,b_i]} f^*(\overrightarrow{z})\, \mathrm{d}\overrightarrow{z} = L.H.S.(3.18), \end{aligned} \tag{3.21}$$

proving the sharpness of (3.18) when n is odd.

ii) When n is even: Notice that $\overline{f}(\overrightarrow{x_0}) = 0$ and any mixed partial of $\overline{f}$ equals zero. Especially we observe that

$$\left\| \frac{\partial^{n+1} \overline{f}(\overrightarrow{z})}{\partial z_i^{n+1}} \right\| = \left(\prod_{j=0}^{n} (n + \alpha_i - j) \right) |z_i - x_{0i}|^{\alpha_i - 1}, \ \ \alpha_i > 1, \tag{3.22}$$

and

$$\left\| \left\| \frac{\partial^{n+1} \overline{f}}{\partial z_i^{n+1}} \right\| \right\|_{\infty} = \left(\prod_{j=0}^{n} (n + \alpha_i - j) \right) \|z_i - x_{0i}\|_{\infty}^{\alpha_i - 1} \tag{3.23}$$

(here $\|z_i - x_{0i}\|_\infty < +\infty$), all $i = 1, \ldots, k$. Hence by plugging $\overline{f}$ into (3.18) we obtain

$$\lim_{\text{all } \alpha_i \to 1} R.H.S.(3.18) = \frac{1}{(n+1)! \prod_{i=1}^k (b_i - a_i)} \cdot \lim_{\text{all } \alpha_i \to 1} \int_{\prod_{i=1}^k [a_i, b_i]} \left[\sum_{i=1}^k |z_i - x_{0i}|^{n+1} \times \times \left(\prod_{j=0}^n (n + \alpha_i - j) \right) \|z_i - x_{0i}\|_\infty^{\alpha_i - 1} \right] \mathrm{d}\overrightarrow{z}$$

$$= \frac{1}{\prod_{i=1}^k (b_i - a_i)} \int_{\prod_{i=1}^k [a_i, b_i]} \left(\sum_{i=1}^k |z_i - x_{0i}|^{n+1} \right) \mathrm{d}\overrightarrow{z}. \tag{3.24}$$

Furthermore,

$$\lim_{\text{all } \alpha_i \to 1} L.H.S.(3.18) = \frac{1}{\prod_{i=1}^k (b_i - a_i)} \lim_{\text{all } \alpha_i \to 1} \int_{\prod_{i=1}^k [a_i, b_i]} \left(\sum_{i=1}^k |z_i - x_{0i}|^{n+\alpha_i} \right) \mathrm{d}\overrightarrow{z} \tag{3.25}$$

$$= \frac{1}{\prod_{i=1}^k (b_i - a_i)} \int_{\prod_{i=1}^k [a_i, b_i]} \left(\sum_{i=1}^k |z_i - x_{0i}|^{n+1} \right) \mathrm{d}\overrightarrow{z}.$$

So we found that

$$\lim_{\text{all } \alpha_i \to 1} R.H.S.(3.18) = \lim_{\text{all } \alpha_i \to 1} L.H.S.(3.18), \tag{3.26}$$

proving the sharpness of (3.18) when n is even. □

Corollary 8 *(to Corollary 7) Let $f \in C^{n+1}([a_1, b_1] \times [a_2, b_2], X)$, $n \in \mathbb{N}$ where $a_1 < b_1$, $a_2 < b_2$; $a_1, a_2, b_1, b_2 \in \mathbb{R}$, and let $\overrightarrow{x_0} = (x_{01}, x_{02}) \in [a_1, b_1] \times [a_2, b_2]$ be fixed. We suppose here that all vector partial derivatives $f_\alpha := \frac{\partial^\alpha f}{\partial z^\alpha}$, where $\alpha = (\alpha_1, \alpha_2)$, $\alpha_1, \alpha_2 \in \mathbb{Z}^+$, $|\alpha| = \alpha_1 + \alpha_2 = j$, $j = 1, \ldots, n$ fulfill $f_\alpha(\overrightarrow{x_0}) = 0$. Then*

$$\left\| \frac{1}{(b_1 - a_1)(b_2 - a_2)} \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(z_1, z_2) \, \mathrm{d}z_1 \, \mathrm{d}z_2 - f(\overrightarrow{x_0}) \right\| \leq$$

$$\sum_{l=0}^{n+1} \frac{\left[(x_{01} - a_1)^{n+2-l} + (b_1 - x_{01})^{n+2-l} \right] \left[(x_{02} - a_2)^{l+1} + (b_2 - x_{02})^{l+1} \right]}{(n+2-l)! \, (l+1)! \, (b_1 - a_1)(b_2 - a_2)} \left\| \left\| \frac{\partial^{n+1} f}{\partial z_1^{n+1-l} \partial z_2^l} \right\| \right\|_\infty . \tag{3.27}$$

Inequality (3.27) *is sharp, exactly the same manner as inequality* (3.18).

Corollary 9 *Let $f \in C^2([a_1, b_1] \times [a_2, b_2], X)$, where $a_1 < b_1$, $a_2 < b_2$; $a_1, a_2, b_1, b_2 \in \mathbb{R}$ and let $\overrightarrow{x_0} = (x_{01}, x_{02}) \in [a_1, b_1] \times [a_2, b_2]$ be fixed. We suppose that $\frac{\partial f}{\partial z_1}(\overrightarrow{x_0}) = \frac{\partial f}{\partial z_2}(\overrightarrow{x_0}) = 0$. Then*

$$\left\| \frac{1}{(b_1 - a_1)(b_2 - a_2)} \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(z_1, z_2) \, \mathrm{d}z_1 dz_2 - f(\overrightarrow{x_0}) \right\| \leq$$

$$\leq \left(\frac{(x_{01} - a_1)^3 + (b_1 - x_{01})^3}{6(b_1 - a_1)} \right) \left\| \left\| \frac{\partial^2 f}{\partial z_1^2} \right\| \right\|_\infty + \left(\frac{(x_{02} - a_2)^3 + (b_2 - x_{02})^3}{6(b_2 - a_2)} \right) \left\| \left\| \frac{\partial^2 f}{\partial z_2^2} \right\| \right\|_\infty +$$

$$+ \left[\frac{\left((x_{01} - a_1)^2 + (b_1 - x_{01})^2 \right) \left((x_{02} - a_2)^2 + (b_2 - x_{02})^2 \right)}{4(b_1 - a_1)(b_2 - a_2)} \right] \left\| \left\| \frac{\partial^2 f}{\partial z_1 \partial z_2} \right\| \right\|_\infty + . \tag{3.28}$$

Inequality (3.28) is sharp; in fact it is attained by

$$f^*(z_1, z_2) = \left((z_1 - x_{01})^2 + (z_2 - x_{02})^2\right) \cdot i_0, \tag{3.29}$$

where $i_0 \in X : \|i_0\| = 1$.

Proof. Application of Corollary 8 when $n = 1$. □

Corollary 10 *Let* $f \in C^2\left(\prod_{i=1}^{3}[a_i, b_i], X\right)$, *where* $a_i < b_i$, $i = 1, 2, 3$; $a_i, b_i \in \mathbb{R}$ *and let* $\overrightarrow{x_0} = (x_{01}, x_{02}, x_{03}) \in \prod_{i=1}^{3}[a_i, b_i]$ *fixed. We suppose here that* $\frac{\partial f}{\partial z_i}(\overrightarrow{x_0}) = 0$; $i = 1, 2, 3$. *Then*

$$\begin{aligned}
&\left\|\frac{1}{\prod_{i=1}^{3}(b_i - a_i)}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(z_1, z_2, z_3)\, dz_1\, dz_2\, dz_3 - f(\overrightarrow{x_0})\right\| \\
&\leq \sum_{i=1}^{3}\frac{(x_{0i} - a_i)^3 + (b_i - x_{0i})^3}{6(b_i - a_i)}\left\|\left\|\frac{\partial^2 f}{\partial z_i^2}\right\|\right\|_\infty \\
&\quad + \sum_{i=1}^{2}\frac{\left((x_{0i} - a_i)^2 + (b_i - x_{0i})^2\right)\left((x_{0,i+1} - a_{i+1})^2 + (b_{i+1} - x_{0,i+1})^2\right)}{4(b_i - a_i)(b_{i+1} - a_{i+1})}\left\|\left\|\frac{\partial^2 f}{\partial z_i \partial z_{i+1}}\right\|\right\|_\infty \\
&\quad + \frac{\left((x_{03} - a_3)^2 + (b_3 - x_{03})^2\right)\left((x_{0,1} - a_1)^2 + (b_1 - x_{01})^2\right)}{4(b_3 - a_3)(b_1 - a_1)}\left\|\left\|\frac{\partial^2 f}{\partial z_3 \partial z_1}\right\|\right\|_\infty .
\end{aligned} \tag{3.30}$$

Inequality (3.30) is sharp and it is attained by

$$f^*(z_1, z_2, z_3) = \left(\sum_{i=1}^{3}(z_i - x_{0i})^2\right) \cdot i_0, \tag{3.31}$$

where $i_0 \in X : \|i_0\| = 1$.

Proof. Use of Corollary 7 for $k = 3$ and $n = 1$. □

We further need

Theorem 11 *Here* $(X, \|\cdot\|)$ *is a Banach space. Let* $f : [a, A] \times [b, B] \times [c, C] \to X$ *be a mapping three times continuously differentiable. Let also* $(x, y, z) \in [a, A] \times [b, B] \times [c, C]$ *be fixed. We define the kernels* $p : [a, A]^2 \to \mathbb{R}$, $q : [b, B]^2 \to \mathbb{R}$, *and* $\theta : [c, C]^2 \to \mathbb{R}$:

$$p(x, s) := \begin{cases} s - a, & s \in [a, x], \\ s - A, & s \in (x, A], \end{cases}$$

$$q(y, t) := \begin{cases} t - b, & t \in [b, y], \\ t - B, & t \in (y, B], \end{cases}$$

and

$$\theta(z, r) := \begin{cases} r - c, & r \in [c, z], \\ r - C, & r \in (z, C]. \end{cases}$$

Then

$$\theta_{1,3} := \int_a^A \int_b^B \int_c^C p(x,s)\, q(y,t)\, \theta(z,t)\, f'''_{s,t,r}(s,t,r)\, \mathrm{d}s\,\mathrm{d}t\,\mathrm{d}r$$
$$= \{(A-a)(B-b)(C-c) f(x,y,z)\} - \left[(B-b)(C-c)\int_a^A f(s,y,z)\,\mathrm{d}s\right.$$
$$\left. + (A-a)(C-c)\int_b^B f(x,t,z)\,\mathrm{d}t + (A-a)(B-b)\int_c^C f(x,y,r)\,\mathrm{d}r\right]$$
$$+ \left[(C-c)\int_a^A\int_b^B f(s,t,z)\,\mathrm{d}sdt + (B-b)\int_a^A\int_c^C f(s,y,r)\,\mathrm{d}sdr\right.$$
$$\left. + (A-a)\int_b^B\int_c^C f(x,t,r)\,\mathrm{d}t\,\mathrm{d}r\right] - \int_a^A\int_b^B\int_c^C f(s,t,r)\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}r =: \theta_{2,3}. \quad (3.32)$$

Proof. Similar to [4], see also [7, p. 82], using integration by parts several times. □

In general we state

Theorem 12 *Here* $(X, \|\cdot\|)$ *is a Banach space. Let* $f : \prod_{i=1}^n [a_i, b_i] \to X$ *be a mapping* n *times continuously differentiable,* $n > 1$*. Let also* $(x_1, \ldots, x_n) \in \prod_{i=1}^n [a_i, b_i]$ *be fixed. We define the kernels* $p_i : [a_i, b_i]^2 \to \mathbb{R}$*:*

$$p_i(x_i, s_i) := \begin{cases} s_i - a_i, & s_i \in [a_i, x_i], \\ s_i - b_i, & s_i \in (x_i, b_i], \end{cases} \quad \textit{for all } i = 1, \ldots, n.$$

Then

$$\theta_{1,n} := \int_{\prod_{i=1}^n [a_i,b_i]} \prod_{i=1}^n p_i(x_i, s_i) \frac{\partial^n f(s_1, \ldots, s_n)}{\partial s_1 \ldots \partial s_n}\,\mathrm{d}s_1 \ldots \mathrm{d}s_n$$
$$= \left(\prod_{i=1}^n (b_i - a_i)\right) f(x_1, \ldots, x_n) - \sum_{i=1}^{\binom{n}{1}} \left(\prod_{\substack{j=1 \\ j \neq i}}^n (b_j - a_j) \int_{a_i}^{b_i} f(x_1, \ldots, s_i, \ldots, x_n)\,\mathrm{d}s_i\right)$$
$$+ \sum_{i=1}^{\binom{n}{2}} \left(\prod_{\substack{k=1 \\ k \neq i,j}}^n (b_k - a_k) \left(\int_{a_i}^{b_i}\int_{a_j}^{b_j} f(x_1, \ldots, s_i, \ldots, s_j, \ldots, x_n)\,\mathrm{d}s_i ds_j\right)\right)_{(l)} - + \ldots$$
$$+ (-1)^{n-1} \sum_{i=1}^{\binom{n}{n-1}} (b_j - a_j) \int_{\prod_{\substack{i=1 \\ i \neq j}}^n [a_i,b_i]} f(s_1, \ldots, x_j, \ldots, s_n)\,\mathrm{d}s_1 \ldots \widehat{\mathrm{d}s_j} \ldots \mathrm{d}s_n$$
$$+ (-1)^n \int_{\prod_{i=1}^n [a_i,b_i]} f(s_1, \ldots, s_n)\,\mathrm{d}s_1 \ldots \mathrm{d}s_n =: \theta_{2,n}. \quad (3.33)$$

The above l *counts all the* (i,j)*'s,* $i < j$*, and* $i, j = 1, \ldots, n$*. Also* $\widehat{\mathrm{d}s_j}$ *means* ds_j *is missing.*

Proof. Similar to Theorem 11, see also [4]. □

We present the following Ostrowski type inequalities:

Theorem 13 *Under the notations and assumptions of Theorem 11, we obtain*

$$\|\theta_{2,3}\| \leq \frac{\left\| \left| f'''_{s,t,r} \right| \right\|_{\infty}}{8} \cdot \left\{ \left((x-a)^2 + (A-x)^2 \right) \cdot \left((y-b)^2 + (B-y)^2 \right) \right.$$

$$\left. \cdot \left((z-c)^2 + (C-z)^2 \right) \right\}, \text{ for all } (x,y,z) \in [a,A] \times [b,B] \times [c,C]. \tag{3.34}$$

Proof. Notice that

$$\|\theta_{2,3}\| = \|\theta_{1,3}\| \leq \left\| \left| f'''_{s,t,r} \right| \right\|_{\infty} \left(\int_a^A |p(x,s)| \, \mathrm{d}s \right) \left(\int_b^B |q(y,t)| \, \mathrm{d}t \right) \left(\int_c^C |\theta(z,r)| \, \mathrm{d}r \right).$$

Also see that

$$\int_a^A |p(x,s)| \, \mathrm{d}s = \frac{1}{2} \left\{ (x-a)^2 + (A-x)^2 \right\},$$

etc. □

The counterpart of the last theorem is

Theorem 14 *Under the notations and assumptions of Theorem 12, we derive*

$$\|\theta_{2,n}\| \leq \frac{\left\| \left| \frac{\partial^n f(s_1,\ldots,s_n)}{\partial s_1 \ldots \partial s_n} \right| \right\|_{\infty}}{2^n} \left\{ \prod_{i=1}^{n} \left[(x_i - a_i)^2 + (b_i - x_i)^2 \right] \right\}, \tag{3.35}$$

for all $(x_1, \ldots, x_n) \in \prod_{i=1}^n [a_i, b_i]$.

Proof. As in Theorem 13. □

It follows

Theorem 15 *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Under the notations and assumptions of Theorem 11 we get*

$$\|\theta_{2,3}\| \leq \frac{\left\| \left| f'''_{s,t,r} \right| \right\|_p}{(q+1)^{\frac{3}{q}}} \cdot \left\{ \left[\left((x-a)^{q+1} + (A-x)^{q+1} \right) \cdot \left((y-b)^{q+1} + (B-y)^{q+1} \right) \right. \right.$$

$$\left. \left. \cdot \left((z-c)^{q+1} + (C-z)^{q+1} \right) \right]^{\frac{1}{q}} \right\}, \text{ for all } (x,y,z) \in [a,A] \times [b,B] \times [c,C]. \tag{3.36}$$

Here

$$\left\| \left| f'''_{s,t,r} \right| \right\|_p := \left(\int_a^A \int_b^B \int_c^C \left\| f'''_{s,t,r}(s,t,r) \right\|^p \mathrm{d}s \, \mathrm{d}t \, \mathrm{d}r \right)^{\frac{1}{p}}.$$

Proof. Notice that

$$\|\theta_{2,3}\| = \|\theta_{1,3}\| = \left\| \int_a^A \int_b^B \int_c^C p(x,s) \, q(y,t) \, \theta(z,r) \, f'''_{s,t,r}(s,t,r) \, \mathrm{d}s \, \mathrm{d}t \, \mathrm{d}r \right\|$$

$$\leq \int_a^A \int_b^B \int_c^C |p(x,s)| \, |q(y,t)| \, |\theta(z,r)| \left\| f'''_{s,t,r}(s,t,r) \right\| \mathrm{d}s \, \mathrm{d}t \, \mathrm{d}r$$

(by Hölder's inequality)

$$\leq \left(\int_a^A \int_b^B \int_c^C \left\| f'''_{s,t,r}(s,t,r) \right\|^p \, \mathrm{d}s \, \mathrm{d}t \, \mathrm{d}r \right)^{\frac{1}{p}}$$

$$\cdot \left(\int_a^A \int_b^B \int_c^C \left(|p(x,s)| \, |q(y,t)| \, |\theta(z,r)| \right)^q \, \mathrm{d}s \, \mathrm{d}t \, \mathrm{d}r \right)^{\frac{1}{q}}$$

$$= \left\| \left\| f'''_{s,t,r} \right\| \right\|_p \left\{ \left(\int_a^A |p(x,s)|^q \, \mathrm{d}s \right) \left(\int_b^B |q(y,t)|^q \, \mathrm{d}t \right) \left(\int_c^C |\theta(z,r)|^q \, \mathrm{d}r \right) \right\}^{\frac{1}{q}}$$

$$= \left\| \left\| f'''_{s,t,r} \right\| \right\|_p \left\{ \left(\frac{(x-a)^{q+1} + (A-x)^{q+1}}{q+1} \right) \cdot \right.$$

$$\left. \cdot \left(\frac{(y-b)^{q+1} + (B-y)^{q+1}}{q+1} \right) \left(\frac{(z-c)^{q+1} + (C-z)^{q+1}}{q+1} \right) \right\}^{\frac{1}{q}}.$$

□

The corresponding general L_p-case follows.

Theorem 16 *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Under the notations and assumptions of Theorem 12 we find*

$$\|\theta_{2,n}\| \leq \frac{\left\| \left\| \frac{\partial^n f(x_1,\ldots,x_n)}{\partial x_1 \ldots \partial x_n} \right\| \right\|_p}{(q+1)^{\frac{n}{q}}} \left\{ \prod_{i=1}^n \left[(x_i - a_i)^{q+1} + (b_i - x_i)^{q+1} \right] \right\}^{\frac{1}{q}}, \tag{3.37}$$

for any $(x_1, \ldots, x_n) \in \prod_{i=1}^n [a_i, b_i]$.

Proof. Similar to Theorem 15. □

Remark 17 *Equalities (3.32) and (3.33) can simplify alot, if for instance we assume in Theorem 11 that theere exists an $(x_0, y_0, z_0) \in [a, A] \times [b, B] \times [c, C]$ such that*

$$f(x_0, \cdot, \cdot) = f(\cdot, y_0, \cdot) = f(\cdot, \cdot, z_0) = 0. \tag{3.38}$$

Also in Theorem 12 we may assume that there exists an $\left(x_1^0, x_2^0, \ldots, x_n^0\right) \in \prod_{i=1}^n [a_i, b_i]$ such that

$$f(x_1^0, x_2, \ldots, x_n) = f\left(x_1, x_2^0, x_3, \ldots, x_n\right) = \cdots = f\left(x_1, \ldots, x_{n-1}, x_n^0\right) = 0, \tag{3.39}$$

for any $(x_1, \ldots, x_n) \in \prod_{i=1}^n [a_i, b_i]$. So in these particular cases we obtain that

$$\theta_{2,3}(x_0, y_0, z_0) = -\int_a^A \int_b^B \int_c^C f(s,t,r) \, \mathrm{d}s \, \mathrm{d}t \, \mathrm{d}r, \tag{3.40}$$

and

$$\theta_{2,n}\left(x_1^0, x_2^0, \ldots, x_n^0\right) = (-1)^n \int_{\prod_{i=1}^n [a_i, b_i]} f(s_1, \ldots, s_n) \, \mathrm{d}s_1 \ldots \mathrm{d}s_n. \tag{3.41}$$

Hence in these cases we obtain

$$\|\theta_{2,3}(x_0, y_0, z_0)\| = \left\| \int_a^A \int_b^B \int_c^C f(s,t,r) \, \mathrm{d}s \, \mathrm{d}t \, \mathrm{d}r \right\|, \tag{3.42}$$

and

$$\left\|\theta_{2,n}\left(x_1^0, x_2^0, \ldots, x_n^0\right)\right\| = \left\|\int_{\prod_{i=1}^n [a_i,b_i]} f(s_1, \ldots, s_n)\, \mathrm{d}s_1 \ldots \mathrm{d}s_n\right\|. \tag{3.43}$$

So (3.34)-(3.37) simplify alot and they become very interesting inequalities.

Theorem 18 *Inequalities (3.34) and (3.35) are sharp.*

Proof. It is enough to prove that (3.35) is sharp. Let $i_0 \in X : \|i_0\| = 1$. Here the optimal function will be

$$f^*(s_1, \ldots, s_n) := \prod_{i=1}^n \left|s_i - x_i^0\right|^\alpha (b_i - a_i)\ i_0, \ \ \alpha > 1, \tag{3.44}$$

where $\left(x_1^0, x_2^0, \ldots, x_n^0\right)$ is fixed in $\prod_{i=1}^n [a_i, b_i]$, $i_0 \in X$, $\|i_0\| = 1$. Notice here that

$$f^*\left(s_1, \ldots, x_j^0, \ldots, s_n\right) = 0, \text{ for all } j = 1, \ldots, n, \text{ and any } (s_1, \ldots, s_n) \in \prod_{i=1}^n [a_i, b_i].$$

Therefore by Remark 17 we have (3.43). We observe that

$$\frac{\partial^n f^*(s_1, \ldots, s_n)}{\partial s_1 \ldots \partial s_n} = \alpha^n \left(\prod_{i=1}^n (b_i - a_i)\right)\left(\prod_{i=1}^n \left|s_i - x_i^0\right|^{\alpha-1} \operatorname{sign}\left(s_i - x_i^0\right)\right) i_0, \tag{3.45}$$

and

$$\left\|\frac{\partial^n f^*(s_1, \ldots, s_n)}{\partial s_1 \ldots \partial s_n}\right\| = \alpha^n \left(\prod_{i=1}^n (b_i - a_i)\right)\left(\prod_{i=1}^n \left|s_i - x_i^0\right|^{\alpha-1}\right). \tag{3.46}$$

Consequently we find

$$\left\|\left\|\frac{\partial^n f^*(s_1, \ldots, s_n)}{\partial s_1 \ldots \partial s_n}\right\|\right\|_\infty = \alpha^n \left(\prod_{i=1}^n (b_i - a_i)\right)\left(\prod_{i=1}^n \left(\max\left(b_i - x_i^0, x_i^0 - a_i\right)\right)^{\alpha-1}\right). \tag{3.47}$$

First we calculate the left-hand side of corresponding inequality (3.35). We have

$$\begin{aligned}
&\left\|\int_{\prod_{i=1}^n [a_i,b_i]} f^*(s_1, \ldots, s_n)\, \mathrm{d}s_1 \ldots \mathrm{d}s_n\right\| \\
&= \int_{\prod_{i=1}^n [a_i,b_i]} \left(\prod_{i=1}^n (b_i - a_i)\right)\left(\prod_{i=1}^n \left|s_i - x_i^0\right|^\alpha\right) \mathrm{d}s_1 \ldots \mathrm{d}s_n \\
&= \left(\prod_{i=1}^n (b_i - a_i)\right) \int_{\prod_{i=1}^n [a_i,b_i]} \left(\prod_{i=1}^n \left|s_i - x_i^0\right|^\alpha\right) \mathrm{d}s_1 \ldots \mathrm{d}s_n \\
&= \left(\prod_{i=1}^n (b_i - a_i)\right) \prod_{i=1}^n \left(\int_{a_i}^{b_i} \left|s_i - x_i^0\right|^\alpha \mathrm{d}s_i\right) \\
&= \left(\prod_{i=1}^n (b_i - a_i)\right) \prod_{i=1}^n \left(\frac{\left(x_i^0 - a_i\right)^{\alpha+1} + \left(b_i - x_i^0\right)^{\alpha+1}}{\alpha + 1}\right).
\end{aligned} \tag{3.48}$$

That is,

$$L.H.S.(3.35) = \frac{\left(\prod_{i=1}^n (b_i - a_i)\right)}{(\alpha+1)^n} \prod_{i=1}^n \left(\left(x_i^0 - a_i\right)^{\alpha+1} + \left(b_i - x_i^0\right)^{\alpha+1}\right). \tag{3.49}$$

And next we see that

$$R.H.S.(3.35) = \frac{\alpha^n \, \mathrm{V}_n \prod_{i=1}^{n} \left(\max\left(b_i - x_i^0, x_i^0 - a_i\right)\right)^{\alpha-1}}{2^n} \cdot \prod_{i=1}^{n} \left(\left(x_i^0 - a_i\right)^2 + \left(b_i - x_i^0\right)^2\right) \quad (3.50)$$

where $\mathrm{V}_n = \left(\prod_{i=1}^{n} (b_i - a_i)\right)$. Now let $\alpha \to 1$. We find

$$\lim_{\alpha \to 1} L.H.S.(3.35) = \frac{\mathrm{V}_n}{2^n} \prod_{i=1}^{n} \left(\left(x_i^0 - a_i\right)^2 + \left(b_i - x_i^0\right)^2\right), \quad (3.51)$$

and

$$\lim_{\alpha \to 1} R.H.S.(3.35) = \frac{\mathrm{V}_n}{2^n} \prod_{i=1}^{n} \left(\left(x_i^0 - a_i\right)^2 + \left(b_i - x_i^0\right)^2\right). \quad (3.52)$$

That is,

$$\lim_{\alpha \to 1} L.H.S.(3.35) = \lim_{\alpha \to 1} R.H.S.(3.35), \quad (3.53)$$

hence proving the sharpness of (3.35). ☐

Remark 19 *Another interesting case for (3.32) and (3.33) is to assume that for specific (x, y, z) ($(x_1, \ldots, x_n)$, respectively) all the marginal integrals of f are equal to zero. Then we get*

$$\theta_{2,3} = (A - a)(B - b)(C - c) f(x, y, z) - \int_a^A \int_b^B \int_c^C f(s, t, r) \, \mathrm{d}s \, \mathrm{d}t \, \mathrm{d}r, \quad (3.54)$$

and

$$\theta_{2,n} = \left(\prod_{i=1}^{n} (b_i - a_i)\right) f(x_1, \ldots, x_n) + (-1)^n \int_{\prod_{i=1}^{n}[a_i, b_i]} f(s_1, \ldots, s_n) \, \mathrm{d}s_1 \ldots \mathrm{d}s_n. \quad (3.55)$$

Hence inequalities (3.34)-(3.37) become again alot simpler.

Next we mention a vector Montgomery identity derived by applying twice integration by parts.

Theorem 20 *([8]) Let $(X, \|\cdot\|)$ be a Banach space and $f \in C^1([a, b], X)$. Let $x \in [a, b]$ be fixed and define*

$$P(x, t) := \begin{cases} \frac{t-a}{b-a}, & a \le t \le x, \\ \frac{t-b}{b-a}, & x < t \le b. \end{cases} \quad (3.56)$$

Then

$$f(x) = \frac{1}{b - a} \int_a^b f(t) \, \mathrm{d}t + \int_a^b P(x, t) f'(t) \, \mathrm{d}t. \quad (3.57)$$

We present a vector multivariate Montgomery identity, see also the real analog in [3]. We have the representation

Theorem 21 *Let $(X, \|\cdot\|)$ be a Banach space and $f \in C^3\left(\prod_{i=1}^3 [a_i, b_i], X\right)$. Let $(x_1, x_2, x_3) \in \prod_{i=1}^3 [a_i, b_i]$. Define the kernels $p_i : [a_i, b_i]^2 \to \mathbb{R}$:*

$$p_i(x_i, s_i) := \begin{cases} s_i - a_i, & s_i \in [a_i, x_i], \\ s_i - b_i, & s_i \in (x_i, b_i], \end{cases} \tag{3.58}$$

for $i = 1, 2, 3$. Then

$$\begin{aligned} f(x_1, x_2, x_3) = \frac{1}{\prod_{i=1}^{3}(b_i - a_i)} \Bigg\{ & \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} f(s_1, s_2, s_3)\, \mathrm{d}s_3\, \mathrm{d}s_2\, \mathrm{d}s_1 \\ & + \sum_{j=1}^{3} \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} p_j(x_j, s_j) \frac{\partial f(s_1, s_2, s_3)}{\partial s_j}\, \mathrm{d}s_3\, \mathrm{d}s_2\, \mathrm{d}s_1 \right) \\ & + \sum_{1 \le j < k \le 3} \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} p_j(x_j, s_j) p_k(x_k, s_k) \frac{\partial^2 f(s_1, s_2, s_3)}{\partial s_k \partial s_j}\, \mathrm{d}s_3\, \mathrm{d}s_2\, \mathrm{d}s_1 \right) \\ & + \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \left(\prod_{i=1}^{3} p_i(x_i, s_i) \right) \frac{\partial^3 f(s_1, s_2, s_3)}{\partial s_3 \partial s_2 \partial s_1}\, \mathrm{d}s_3\, \mathrm{d}s_2\, \mathrm{d}s_1 \Bigg\}. \end{aligned} \tag{3.59}$$

Proof. Multiple use of (3.57), see also [3] and [6, p. 16]. □

A generalization of Theorem 21 to any $n \in \mathbb{N}$ follows.

Theorem 22 *Let $(X, \|\cdot\|)$ be a Banach space and $f \in C^n(D_n, X)$, where $D_n = \prod_{i=1}^n [a_i, b_i]$. Let $(x_1, \ldots, x_n) \in D_n$. Define the kernels $p_i : [a_i, b_i]^2 \to \mathbb{R}$:*

$$p_i(x_i, s_i) := \begin{cases} s_i - a_i, & s_i \in [a_i, x_i], \\ s_i - b_i, & s_i \in (x_i, b_i], \end{cases} \tag{3.60}$$

for $i = 1, 2, \ldots, n$. Then

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) = \frac{1}{\prod_{i=1}^{n}(b_i - a_i)} \Bigg\{ & \int_{D_n} f(s_1, s_2, \ldots, s_n)\, \mathrm{d}^n\vec{s} \\ & + \sum_{j=1}^{n} \left(\int_{D_n} p_j(x_j, s_j) \frac{\partial f(s_1, s_2, \ldots, s_n)}{\partial s_j}\, \mathrm{d}^n\vec{s} \right) \\ & + \sum_{1 \le j < k \le n} \left(\int_{D_n} p_j(x_j, s_j) p_k(x_k, s_k) \frac{\partial^2 f(s_1, s_2, \ldots, s_n)}{\partial s_k \partial s_j}\, \mathrm{d}^n\vec{s} \right) \\ & + \sum_{1 \le j < k < r \le n} \left(\int_{D_n} p_j(x_j, s_j)\, p_k(x_k, s_k)\, p_r(x_r, s_r) \frac{\partial^3 f(s_1, \ldots, s_n)}{\partial s_r\, \partial s_k\, \partial s_j}\, \mathrm{d}^n\vec{s} \right) \\ & + \cdots + \sum_{l=1}^{n} \left(\int_{D_n} p_1(x_1, s_1) \ldots \widehat{p_l(x_l, s_l)} \ldots p_n(x_n, s_n) \frac{\partial^{n-1} f(s_1, \ldots, s_n)}{\partial s_n \ldots \widehat{\partial s_l} \ldots \partial s_1}\, \mathrm{d}^n\vec{s} \right) \\ & + \int_{D_n} \left(\prod_{i=1}^{n} p_i(x_i, s_i) \right) \frac{\partial^n f(s_1, \ldots, s_n)}{\partial s_n \ldots \partial s_1}\, \mathrm{d}^n\vec{s} \Bigg\}. \end{aligned} \tag{3.61}$$

In the above, $\widehat{p_l(x_l, s_l)}$ and $\widehat{\partial s_l}$ mean that $p_l(x_l, s_l)$ and ∂s_l are missing, respectively.

Proof. Similar to Theorem 21, see also [3]. □

Next we obtain the following Ostrowski type inequalities:

Theorem 23 *Let $f : \prod_{i=1}^{3}[a_i,b_i] \to X$ as in Theorem 21. Then*

$$\left\| f(x_1,x_2,x_3) - \frac{1}{\prod_{i=1}^{3}(b_i-a_i)} \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(s_1,s_2,s_3)\,\mathrm{d}s_3\,\mathrm{d}s_2 ds_1 \right\| \le \frac{M_{1,3}+M_{2,3}+M_{3,3}}{\prod_{i=1}^{3}(b_i-a_i)}. \tag{3.62}$$

Here we have

$$M_{1,3} := \min \begin{cases} \sum_{j=1}^{3} \left\| \frac{\partial f}{\partial s_j} \right\|_{\infty} \left(\prod_{\substack{i=1\\ i\neq j}}^{3} (b_i-a_i) \right) \left(\frac{(x_j-a_j)^2+(b_j-x_j)^2}{2} \right), \\ \sum_{j=1}^{3} \left\| \frac{\partial f}{\partial s_j} \right\|_{p_j} \left(\prod_{\substack{i=1\\ i\neq j}}^{3} (b_i-a_i)^{\frac{1}{q_j}} \right) \left[\frac{(x_j-a_j)^{q_j+1}+(b_j-x_j)^{q_j+1}}{q_j+1} \right]^{\frac{1}{q_j}} \\ \quad \text{when } p_j,q_j>1: \frac{1}{p_j}+\frac{1}{q_j}=1, \text{for } j=1,2,3; \\ \sum_{j=1}^{3} \left\| \frac{\partial f}{\partial s_j} \right\|_{1} \left(\frac{b_j-a_j}{2} + \left| x_j - \frac{a_j+b_j}{2} \right| \right). \end{cases} \tag{3.63}$$

Also we have

$$M_{2,3} := \min \begin{cases} \sum_{1\le j<k\le 3} \left\| \frac{\partial^2 f}{\partial s_k\,\partial s_j} \right\|_{\infty} \left(\prod_{i\neq j,k} (b_i-a_i) \right) \frac{\left((x_j-a_j)^2+(b_j-x_j)^2\right)\left((x_k-a_k)^2+(b_k-x_k)^2\right)}{4}; \\ \sum_{1\le j<k\le 3} \left\| \frac{\partial^2 f}{\partial s_k\,\partial s_j} \right\|_{p_{kj}} \left(\prod_{i\neq j,k} (b_i-a_i)^{\frac{1}{q_{kj}}} \right) \left[\frac{(x_j-a_j)^{q_{kj}+1}+(b_j-x_j)^{q_{kj}+1}}{q_{kj}+1} \right]^{\frac{1}{q_{kj}}} \cdot \\ \qquad \cdot \left[\frac{(x_k-a_k)^{q_{kj}+1}+(b_k-x_k)^{q_{kj}+1}}{q_{kj}+1} \right]^{\frac{1}{q_{kj}}}, \\ \quad \text{when } p_{kj},q_{kj}>1: \frac{1}{p_{kj}}+\frac{1}{q_{kj}}=1, \text{for } j,k\in\{1,2,3\}; \\ \sum_{1\le j<k\le 3} \left\| \frac{\partial^2 f}{\partial s_k\,\partial s_j} \right\|_{1} \left(\frac{b_j-a_j}{2} + \left| x_j - \frac{a_j+b_j}{2} \right| \right) \left(\frac{b_k-a_k}{2} + \left| x_k - \frac{a_k+b_k}{2} \right| \right). \end{cases} \tag{3.64}$$

And finally

$$M_{3,3} := \min \begin{cases} \left\| \frac{\partial^3 f(s_1,s_2,s_3)}{\partial s_3\,\partial s_2\,\partial s_1} \right\|_{\infty} \frac{1}{8} \prod_{j=1}^{3} \left((x_j-a_j)^2+(b_j-x_j)^2 \right); \\ \left\| \frac{\partial^3 f(s_1,s_2,s_3)}{\partial s_3\,\partial s_2\,\partial s_1} \right\|_{p} \prod_{i=1}^{3} \left(\frac{(x_j-a_j)^{q+1}+(b_j-x_j)^{q+1}}{q+1} \right)^{\frac{1}{q}}, \\ \quad \text{when } p,q>1: \frac{1}{p}+\frac{1}{q}=1; \\ \left\| \frac{\partial^3 f(s_1,s_2,s_3)}{\partial s_3\,\partial s_2\,\partial s_1} \right\|_{1} \prod_{j=1}^{3} \left(\frac{b_j-a_j}{2} + \left| x_j - \frac{a_j+b_j}{2} \right| \right). \end{cases} \tag{3.65}$$

Inequality (3.62) is true for any $(x_1,x_2,x_3) \in \prod_{i=1}^{3}[a_i,b_i]$, where $\|\cdot\|_p$ ($1 \le p \le \infty$) are the usual L_p-norms on $\prod_{i=1}^{3}[a_i,b_i]$.

Proof. We have by (3.59) that

$$\left\| f(x_1,x_2,x_3) - \frac{1}{\prod_{i=1}^{3}(b_i-a_i)} \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(s_1,s_2,s_3)\,\mathrm{d}s_3\,\mathrm{d}s_2\,\mathrm{d}s_1 \right\|$$

$$\leq \frac{1}{\prod_{i=1}^{3}(b_i-a_i)} \left\{ \sum_{j=1}^{3} \left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} |p_j(x_j,s_j)| \left\| \frac{\partial f(s_1,s_2,s_3)}{\partial s_j} \right\| \mathrm{d}s_3\,\mathrm{d}s_2\,\mathrm{d}s_1 \right) \right.$$

$$+ \sum_{1\leq j<k\leq 3} \left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} |p_j(x_j,s_j)|\,|p_k(x_k,s_k)| \left\| \frac{\partial^2 f(s_1,s_2,s_3)}{\partial s_k\,\partial s_j} \right\| \mathrm{d}s_3\,\mathrm{d}s_2\,\mathrm{d}s_1 \right)$$

$$\left. + \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} \left(\prod_{i=1}^{3} |p_i(x_i,s_i)| \right) \left\| \frac{\partial^3 f(s_1,s_2,s_3)}{\partial s_3\,\partial s_2\,\partial s_1} \right\| \mathrm{d}s_3\,\mathrm{d}s_2\,\mathrm{d}s_1 \right\}.$$

We notice the following ($j = 1,2,3$)

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} |p_j(x_j,s_j)| \left\| \frac{\partial f(s_1,s_2,s_3)}{\partial s_j} \right\| \mathrm{d}s_3\,\mathrm{d}s_2\,\mathrm{d}s_1 \leq$$

$$\leq \min \begin{cases} \left\| \left\| \frac{\partial f}{\partial s_j} \right\| \right\|_\infty \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} |p_j(x_j,s_j)|\,\mathrm{d}s_3\,\mathrm{d}s_2\,\mathrm{d}s_1, \\ \left\| \left\| \frac{\partial f}{\partial s_j} \right\| \right\|_{p_j} \left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} |p_j(x_j,s_j)|^{q_j}\,\mathrm{d}s_3\,\mathrm{d}s_2\,\mathrm{d}s_1 \right)^{\frac{1}{q_j}}, \\ \quad p_j,q_j>1: \frac{1}{p_j}+\frac{1}{q_j}=1; \\ \left\| \left\| \frac{\partial f}{\partial s_j} \right\| \right\|_1 \sup_{s_j\in[a_j,b_j]} |p_j(x_j,s_j)|. \end{cases}$$

Furthermore we see that

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} |p_j(x_j,s_j)|\,\mathrm{d}s_3\,\mathrm{d}s_2\,\mathrm{d}s_1 = (b_1-a_1)\,(\widehat{b_j-a_j})\,(b_3-a_3)\,\frac{(x_j-a_2)^2+(b_j-x_j)^2}{2},$$

where $\widehat{b_j-a_j}$ means (b_j-a_j) is missing, for $j=1,2,3$. Also we find

$$\left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} |p_j(x_j,s_j)|^{q_j}\,\mathrm{d}s_3\,\mathrm{d}s_2\,\mathrm{d}s_1 \right)^{\frac{1}{q_j}}$$

$$= (b_1-a_1)^{\frac{1}{q_j}}\,(\widehat{b_j-a_j})^{\frac{1}{q_j}}\,(b_3-a_3)^{\frac{1}{q_j}} \left(\frac{(x_j-a_2)^{q_j+1}+(b_j-x_j)^{q_j+1}}{q_j+1} \right)^{\frac{1}{q_j}},$$

where $\left(\widehat{b_j-a_j}\right)^{\frac{1}{q_j}}$ means that $(b_j-a_j)^{\frac{1}{q_j}}$ is missing, for $j=1,2,3$. Also

$$\sup_{s_j\in[a_j,b_j]} |p_j(x_j,s_j)| = \max\{x_j-a_j, b_j-x_j\} = \frac{b_j-a_j}{2} + \left| x_j - \frac{a_j+b_j}{2} \right|,$$

for $j=1,2,3$.

Putting things together we get

$$\sum_{j=1}^{3}\left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}|p_j(x_j,s_j)|\left\|\frac{\partial f(s_1,s_2,s_3)}{\partial s_j}\right\|\mathrm{d}s_3\,\mathrm{d}s_2\,\mathrm{d}s_1\right)\leq$$

$$\min\begin{cases}\sum_{j=1}^{3}\left\{\left\|\left\|\frac{\partial f}{\partial s_j}\right\|\right\|_{\infty}\left(\prod_{\substack{i=1\\ i\neq j}}^{3}(b_i-a_i)\right)\left(\frac{(x_j-a_j)^2+(b_j-x_j)^2}{2}\right)\right\};\\ \sum_{j=1}^{3}\left\{\left\|\left\|\frac{\partial f}{\partial s_j}\right\|\right\|_{p_j}\left(\prod_{\substack{i=1\\ i\neq j}}^{3}(b_i-a_i)^{\frac{1}{q_j}}\right)\left[\frac{(x_j-a_j)^{q_j+1}+(b_j-x_j)^{q_j+1}}{q_j+1}\right]^{\frac{1}{q_j}}\right\},\\ \qquad\text{when } p_j,q_j>1:\frac{1}{p_j}+\frac{1}{q_j}=1,\text{ for } j=1,2,3;\\ \sum_{j=1}^{3}\left\{\left\|\left\|\frac{\partial f}{\partial s_j}\right\|\right\|_{1}\left(\frac{b_j-a_j}{2}+\left|x_j-\frac{a_j+b_j}{2}\right|\right)\right\}.\end{cases}\tag{3.66}$$

By similar work, next we find that

$$\sum_{1\leq j<k\leq 3}\left(\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}|p_j(x_j,s_j)|\,|p_k(x_k,s_k)|\left\|\frac{\partial^2 f(s_1,s_2,s_3)}{\partial s_k\,\partial s_j}\right\|\mathrm{d}s_3\,\mathrm{d}s_2\,\mathrm{d}s_1\right)\leq$$

$$\min\begin{cases}\sum_{1\leq j<k\leq 3}\left\{\left\|\left\|\frac{\partial^2 f}{\partial s_k\,\partial s_j}\right\|\right\|_{\infty}\left(\prod_{i\neq j,k}(b_i-a_i)\right)\frac{\left((x_j-a_j)^2+(b_j-x_j)^2\right)\left((x_k-a_k)^2+(b_k-x_k)^2\right)}{4}\right\};\\ \sum_{1\leq j<k\leq 3}\left\{\left\|\left\|\frac{\partial^2 f}{\partial s_k\,\partial s_j}\right\|\right\|_{p_{kj}}\left(\prod_{i\neq j,k}(b_i-a_i)^{\frac{1}{q_{kj}}}\right)\left[\frac{(x_j-a_j)^{q_{kj}+1}+(b_j-x_j)^{q_{kj}+1}}{q_{kj}+1}\right]^{\frac{1}{q_{kj}}}\cdot\right.\\ \left.\cdot\left[\frac{(x_k-a_k)^{q_{kj}+1}+(b_k-x_k)^{q_{kj}+1}}{q_{kj}+1}\right]^{\frac{1}{q_{kj}}}\right\},\ \text{when } p_{kj},q_{kj}>1:\frac{1}{p_{kj}}+\frac{1}{q_{kj}}=1,\text{ for } j,k\in\{1,2,3\};\\ \sum_{1\leq j<k\leq 3}\left\{\left\|\left\|\frac{\partial^2 f}{\partial s_k\,\partial s_j}\right\|\right\|_{1}\left(\frac{b_j-a_j}{2}+\left|x_j-\frac{a_j+b_j}{2}\right|\right)\left(\frac{b_k-a_k}{2}+\left|x_k-\frac{a_k+b_k}{2}\right|\right)\right\}.\end{cases}\tag{3.67}$$

Finally we get that

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\left(\prod_{i=1}^{3}|p_i(x_i,s_i)|\right)\left\|\frac{\partial^3 f(s_1,s_2,s_3)}{\partial s_3\,\partial s_2\,\partial s_1}\right\|\mathrm{d}s_3\,\mathrm{d}s_2\,\mathrm{d}s_1\leq$$

$$\min\begin{cases}\left\|\left\|\frac{\partial^3 f(s_1,s_2,s_3)}{\partial s_3\,\partial s_2\,\partial s_1}\right\|\right\|_{\infty}\frac{1}{8}\prod_{j=1}^{3}\left((x_j-a_j)^2+(b_j-x_j)^2\right);\\ \left\|\left\|\frac{\partial^3 f(s_1,s_2,s_3)}{\partial s_3\,\partial s_2\,\partial s_1}\right\|\right\|_{p}\prod_{j=1}^{3}\left(\frac{(x_j-a_j)^{q+1}+(b_j-x_j)^{q+1}}{q+1}\right)^{\frac{1}{q}},\\ \qquad\text{when } p,q>1:\frac{1}{p}+\frac{1}{q}=1;\\ \left\|\left\|\frac{\partial^3 f(s_1,s_2,s_3)}{\partial s_3\,\partial s_2\,\partial s_1}\right\|\right\|_{1}\prod_{j=1}^{3}\left(\frac{b_j-a_j}{2}+\left|x_j-\frac{a_j+b_j}{2}\right|\right).\end{cases}\tag{3.68}$$

Taking into account (3.66), (3.67), (3.68) we have completed the proof of (3.62). □

A generalization of Theorem 23 follows.

Theorem 24 *Let* $f:\prod_{i=1}^{n}[a_i,b_i]\to X$ *as in Theorem 22,* $n\in\mathbb{N}$*, and* $(x_1,\dots,x_n)\in\prod_{i=1}^{n}[a_i,b_i]$*. Here* $\|\cdot\|_p$ *(*$1\le p\le\infty$*) is the usual* L_p*-norm on* $\prod_{i=1}^{n}[a_i,b_i]$*. Then*

$$\left\|f(x_1,x_2,\dots,x_n)-\frac{1}{\prod\limits_{i=1}^{n}(b_i-a_i)}\int_{\prod_{i=1}^{n}[a_i,b_i]}f(s_1,s_2,\dots,s_n)\,\mathrm{d}s_n\dots\mathrm{d}s_1\right\|\le\frac{\sum\limits_{i=1}^{n}M_{1,n}}{\prod\limits_{i=1}^{n}(b_i-a_i)}. \tag{3.69}$$

Here we have

$$M_{1,n}:=\min\begin{cases}\sum\limits_{j=1}^{n}\left\{\left\|\left\|\frac{\partial f}{\partial s_j}\right\|\right\|_\infty\left(\prod\limits_{\substack{i=1\\ i\ne j}}^{n}(b_i-a_i)\right)\left(\frac{(x_j-a_j)^2+(b_j-x_j)^2}{2}\right)\right\};\\ \sum\limits_{j=1}^{n}\left\{\left\|\left\|\frac{\partial f}{\partial s_j}\right\|\right\|_p\left(\prod\limits_{\substack{i=1\\ i\ne j}}^{n}(b_i-a_i)^{\frac1q}\right)\left[\frac{(x_j-a_j)^{q+1}+(b_j-x_j)^{q+1}}{q+1}\right]^{\frac1q}\right\},\\ \quad\text{when } p,q>1:\frac1p+\frac1q=1, \text{for } j=1,2,\dots,n;\\ \sum\limits_{j=1}^{n}\left\{\left\|\left\|\frac{\partial f}{\partial s_j}\right\|\right\|_1\left(\frac{b_j-a_j}{2}+\left|x_j-\frac{a_j+b_j}{2}\right|\right)\right\};\end{cases} \tag{3.70}$$

and, with $j,k\in\{1,2,\dots,n\}$,

$$M_{2,n}:=\min\begin{cases}\sum\limits_{j<k}\left\{\left\|\left\|\frac{\partial^2 f}{\partial s_k\partial s_j}\right\|\right\|_\infty\left(\prod\limits_{i\ne j,k}(b_i-a_i)\right)\frac{\left((x_j-a_j)^2+(b_j-x_j)^2\right)\left((x_k-a_k)^2+(b_k-x_k)^2\right)}{4}\right\};\\ \sum\limits_{j<k}\left\{\left\|\left\|\frac{\partial^2 f}{\partial s_k\partial s_j}\right\|\right\|_p\left(\prod\limits_{i\ne j,k}(b_i-a_i)^{\frac1q}\right)\left[\frac{(x_j-a_j)^{q+1}+(b_j-x_j)^{q+1}}{q+1}\right]^{\frac1q}\cdot\right.\\ \qquad\left.\cdot\left[\frac{(x_k-a_k)^{q+1}+(b_k-x_k)^{q+1}}{q+1}\right]^{\frac1q}\right\},\quad\text{when } p,q>1:\frac1p+\frac1q=1;\\ \sum\limits_{j<k}\left\{\left\|\left\|\frac{\partial^2 f}{\partial s_k\partial s_j}\right\|\right\|_1\left(\frac{b_j-a_j}{2}+\left|x_j-\frac{a_j+b_j}{2}\right|\right)\left(\frac{b_k-a_k}{2}+\left|x_k-\frac{a_k+b_k}{2}\right|\right)\right\};\end{cases} \tag{3.71}$$

and for $j,k,r\in\{1,\dots,n\}$,

$$M_{3,n}:=\min\begin{cases}\sum\limits_{j<k<r}\left\|\left\|\frac{\partial^3 f}{\partial s_r\partial s_k\partial s_j}\right\|\right\|_\infty\left[\prod\limits_{i\ne j,k,r}(b_i-a_i)\right]\prod\limits_{m\in\{j,k,r\}}\frac{(x_m-a_m)^2+(b_m-x_m)^2}{2};\\ \sum\limits_{j<k<r}\left\|\left\|\frac{\partial^3 f}{\partial s_r\partial s_k\partial s_j}\right\|\right\|_p\left[\prod\limits_{i\ne j,k,r}(b_i-a_i)^{\frac1q}\right]\prod\limits_{m\in\{j,k,r\}}\left(\frac{(x_m-a_m)^{q+1}+(b_m-x_m)^{q+1}}{q+1}\right)^{\frac1q},\\ \qquad\text{when } p,q>1:\frac1p+\frac1q=1;\\ \sum\limits_{j<k<r}\left\|\left\|\frac{\partial^3 f}{\partial s_r\partial s_k\partial s_j}\right\|\right\|_1\prod\limits_{m\in\{j,k,r\}}\left(\frac{b_m-a_m}{2}+\left|x_m-\frac{a_m+b_m}{2}\right|\right);\end{cases} \tag{3.72}$$

and for $l = 1, \dots, n$,

$$M_{n-1,n} := \min \begin{cases} \sum_{l=1}^{n} \left\| \left\| \frac{\partial^{n-1} f}{\partial s_n \dots \widehat{\partial s_l} \dots \partial s_1} \right\| \right\|_\infty (b_l - a_l) \prod_{\substack{m=1 \\ m \neq l}}^{n} \frac{(x_m - a_m)^2 + (b_m - x_m)^2}{2}, \\ \sum_{l=1}^{n} \left\| \left\| \frac{\partial^{n-1} f}{\partial s_n \dots \widehat{\partial s_l} \dots \partial s_1} \right\| \right\|_p (b_l - a_l)^{\frac{1}{q}} \prod_{\substack{m=1 \\ m \neq l}}^{n} \left[\frac{(x_m - a_m)^{q+1} + (b_m - x_m)^{q+1}}{q+1} \right]^{\frac{1}{q}}, \\ \qquad \text{when } p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1; \\ \sum_{l=1}^{n} \left\| \left\| \frac{\partial^{n-1} f}{\partial s_n \dots \widehat{\partial s_l} \dots \partial s_1} \right\| \right\|_1 \prod_{\substack{m=1 \\ m \neq l}}^{n} \left(\frac{b_m - a_m}{2} + \left| x_m - \frac{a_m + b_m}{2} \right| \right). \end{cases} \tag{3.73}$$

Finally we have

$$M_{n,n} := \min \begin{cases} \left\| \left\| \frac{\partial^n f}{\partial s_n \dots \partial s_1} \right\| \right\|_\infty \prod_{j=1}^{n} \frac{(x_j - a_j)^2 + (b_j - x_j)^2}{2}; \\ \left\| \left\| \frac{\partial^n f}{\partial s_n \dots \partial s_1} \right\| \right\|_p \sum_{j=1}^{n} \left[\frac{(x_j - a_j)^{q+1} + (b_j - x_j)^{q+1}}{q+1} \right]^{\frac{1}{q}}, \\ \qquad \text{when } p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1; \\ \left\| \left\| \frac{\partial^n f}{\partial s_n \dots \partial s_1} \right\| \right\|_1 \prod_{j=1}^{n} \left(\frac{b_j - a_j}{2} + \left| x_j - \frac{a_j + b_j}{2} \right| \right). \end{cases} \tag{3.74}$$

Proof. Similar to Theorem 23. □

We need from [8]

Lemma 25 *Let* $f \in C^n([a,b], X)$, $n \in \mathbb{N}$, $(X, \|\cdot\|)$ *a Banach space,* $x \in [a,b]$. *Then*

$$f(x) = \frac{n \int_a^b f(y)\,dy}{b-a} + \sum_{k=1}^{n-1} \frac{n-k}{k!} \left(\frac{f^{(k-1)}(b)(x-b)^k - f^{(k-1)}(a)(x-a)^k}{b-a} \right)$$

$$+ \frac{1}{(n-1)!\,(b-a)} \int_a^b \left(\int_y^x (x-t)^{n-1} f^{(n)}(t)\,dt \right) dy. \tag{3.75}$$

We present the representation result

Theorem 26 *Let* $(X, \|\cdot\|)$ *be a Banach space and* $f \in C^n\Big(\prod_{j=1}^{m} [a_j, b_j], X\Big)$, $m, n \in \mathbb{N}$. *Let also* $(x_1, \dots, x_m) \in \prod_{j=1}^{m} [a_j, b_j]$ *be fixed. Then*

$$f(x_1, \dots, x_n) = \frac{n^m}{\prod_{j=1}^{m} (b_j - a_j)} \int_{\prod_{j=1}^{m} [a_j, b_j]} f(s_1, \dots, s_m)\,ds_1 \dots ds_m + \sum_{i=1}^{m} T_i, \tag{3.76}$$

where for $i = 1, \ldots, m$ we put

$$\begin{aligned}
T_i &:= T_i(x_i, \ldots, x_m) \\
&:= \frac{n^{i-1}}{\prod_{j=1}^{i} (b_j - a_j)} \sum_{k=1}^{n-1} \frac{n-k}{k!} \int_{\prod_{j=1}^{i-1} [a_j, b_j]} \Bigg(\frac{\partial^{k-1} f}{\partial x_i^{k-1}} (s_1, s_2, \ldots, s_{i-1}, b_i, x_{i+1}, \ldots, x_m) (x_i - b_i)^k \\
&\qquad - \frac{\partial^{k-1} f}{\partial x_i^{k-1}} (s_1, s_2, \ldots, s_{i-1}, a_i, x_{i+1}, \ldots, x_m) (x_i - a_i)^k \Bigg) \mathrm{d}s_1 \, \mathrm{d}s_2 \ldots \mathrm{d}s_{i-1} \\
&\quad + \frac{n^{i-1}}{(n-1)! \prod_{j=1}^{i} (b_j - a_j)} \int_{\prod_{j=1}^{i} [a_j, b_j]} \Bigg(\int_{s_i}^{x_i} (x_i - t_i)^{n-1} \times \\
&\qquad \times \frac{\partial^n f}{\partial x_i^n} (s_1, s_2, s_3, \ldots, s_{i-1}, t_i, x_{i+1}, \ldots, x_m) \, \mathrm{d}t_i \Bigg) \mathrm{d}s_i \, \mathrm{d}s_{i-1} \ldots \mathrm{d}s_1.
\end{aligned} \tag{3.77}$$

When $n = 1$, the $\sum_{k=1}^{n-1}$ in (3.77) is zero. (For the real analog see [5] and [7, p. 367].)

Proof. Here $(X, \|\cdot\|)$ is a Banach space and $f \in C^n\left(\prod_{j=1}^{m} [a_j, b_j], X\right)$, $m, n \in \mathbb{N}$. Hence by Lemma 25 we have

$$f(x_1, \ldots, x_m) = \frac{n}{b_1 - a_1} \int_{a_1}^{b_1} f(s_1, x_2, \ldots, x_m) \, \mathrm{d}s_1 + T_1(x_1, \ldots, x_m), \tag{3.78}$$

where

$$\begin{aligned}
&T_1(x_1, \ldots, x_m) = \\
&\frac{1}{b_1 - a_1} \sum_{k=1}^{n-1} \frac{n-k}{k!} \left(\frac{\partial^{k-1} f(b_1, x_2, \ldots, x_m)}{\partial x_1^{k-1}} (x_1 - b_1)^k - \frac{\partial^{k-1} f(a_1, x_2, \ldots, x_m)}{\partial x_1^{k-1}} (x_1 - a_1)^k \right) \\
&+ \frac{1}{(n-1)! (b_1 - a_1)} \int_{a_1}^{b_1} \left(\int_{s_1}^{x_1} (x_1 - t_1)^{n-1} \frac{\partial^n f}{\partial x_1^n} (t_1, x_2, \ldots, x_m) \, \mathrm{d}t_1 \right) \mathrm{d}s_1.
\end{aligned} \tag{3.79}$$

But it holds

$$\begin{aligned}
f(s_1, x_2, \ldots, x_m) &= \frac{n}{b_2 - a_2} \int_{a_2}^{b_2} f(s_1, s_2, x_3 \ldots, x_m) \, \mathrm{d}s_2 + \frac{1}{b_2 - a_2} \sum_{k=1}^{n-1} \frac{n-k}{k!} \times \\
&\times \left(\frac{\partial^{k-1} f(s_1, b_2, x_3 \ldots, x_m)}{\partial x_2^{k-1}} (x_2 - b_2)^k - \frac{\partial^{k-1} f(s_1, a_2, x_3, \ldots, x_m)}{\partial x_2^{k-1}} (x_2 - a_2)^k \right) \\
&+ \frac{1}{(n-1)! (b_2 - a_2)} \int_{a_2}^{b_2} \left(\int_{s_2}^{x_2} (x_2 - t_2)^{n-1} \frac{\partial^n f}{\partial x_2^n} (s_1, t_2, x_3, \ldots, x_m) \, \mathrm{d}t_2 \right) ds_2.
\end{aligned} \tag{3.80}$$

Combining (3.78) and (3.80) we obtain

$$f(x_1,\ldots,x_m)=\frac{n^2}{(b_1-a_1)(b_2-a_2)}\int_{a_1}^{b_1}\int_{a_2}^{b_2}f(s_1,s_2,x_3,\ldots,x_m)\,\mathrm{d}s_2\,\mathrm{d}s_1$$

$$+T_2(x_2,x_3,\ldots,x_m)+T_1(x_1,\ldots,x_m), \tag{3.81}$$

where

$$T_2(x_2,x_3,..,x_m)=\frac{n}{(b_1-a_1)(b_2-a_2)}\sum_{k=1}^{n-1}\left(\frac{n-k}{k!}\right)\cdot$$
$$\cdot\int_{a_1}^{b_1}\left(\frac{\partial^{k-1}f(s_1,b_2,x_3\ldots,x_m)}{\partial x_2^{k-1}}(x_2-b_2)^k-\frac{\partial^{k-1}f(s_1,a_2,x_3,\ldots,x_m)}{\partial x_2^{k-1}}(x_2-a_2)^k\right)\mathrm{d}s_1$$
$$+\frac{n}{(n-1)!\,(b_1-a_1)(b_2-a_2)}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{s_2}^{x_2}(x_2-t_2)^{n-1}\frac{\partial^n f}{\partial x_2^n}(s_1,t_2,x_3,\ldots,x_m)\,\mathrm{d}t_2\,\mathrm{d}s_2\,\mathrm{d}s_1.$$

Next we see that

$$f(s_1,s_2,x_3,\ldots,x_m)=\frac{n}{b_3-a_3}\int_{a_3}^{b_3}f(s_1,s_2,s_3,x_4\ldots,x_m)\,\mathrm{d}s_3+\frac{1}{(b_3-a_3)}\sum_{k=1}^{n-1}\left(\frac{n-k}{k!}\right)\cdot$$
$$\cdot\left(\frac{\partial^{k-1}f(s_1,s_2,b_3,x_4,\ldots,x_m)}{\partial x_3^{k-1}}(x_3-b_3)^k-\frac{\partial^{k-1}f(s_1,s_2,a_3,x_4,\ldots,x_m)}{\partial x_3^{k-1}}(x_3-a_3)^k\right)$$
$$+\frac{1}{(n-1)!\,(b_3-a_3)}\int_{a_3}^{b_3}\left(\int_{s_3}^{x_3}(x_3-t_3)^{n-1}\frac{\partial^n f}{\partial x_3^n}(s_1,s_2,t_3,x_4,\ldots,x_m)\,\mathrm{d}t_3\right)\mathrm{d}s_3. \tag{3.82}$$

Combining (3.81) and (3.82) we get

$$f(x_1,\ldots,x_m)=\frac{n^3}{\prod_{i=1}^{3}(b_i-a_i)}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}f(s_1,s_2,s_3,x_4\ldots,x_m)\,\mathrm{d}s_3\,\mathrm{d}s_2\,\mathrm{d}s_1$$
$$+T_3(x_3,x_4,\ldots,x_m)+T_1+T_2, \tag{3.83}$$

where

$$T_3(x_3,x_4,\ldots,x_m)=\frac{n^2}{\prod_{i=1}^{3}(b_i-a_i)}\sum_{k=1}^{n-1}\frac{n-k}{k!}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\Bigg[\frac{\partial^{k-1}f(s_1,s_2,b_3,x_4,\ldots,x_m)}{\partial x_3^{k-1}}(x_3-b_3)^k$$
$$-\frac{\partial^{k-1}f(s_1,s_2,a_3,x_4,\ldots,x_m)}{\partial x_3^{k-1}}(x_3-a_3)^k\Bigg]\mathrm{d}s_2\,\mathrm{d}s_1$$
$$+\frac{n^2}{(n-1)!\prod_{i=1}^{3}(b_i-a_i)}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\int_{s_3}^{x_3}(x_3-t_3)^{n-1}\times$$
$$\times\frac{\partial^n f}{\partial x_3^n}(s_1,s_2,t_3,x_4,\ldots,x_m)\,\mathrm{d}t_3\,\mathrm{d}s_3\,\mathrm{d}s_2\,\mathrm{d}s_1. \tag{3.84}$$

We also observe that

$$f(s_1,s_2,s_3,x_4,\dots,x_m)=\frac{n}{b_4-a_4}\int_{a_4}^{b_4} f(s_1,s_2,s_3,s_4,x_5,\dots,x_m)\,\mathrm{d}s_4$$
$$+\frac{1}{(b_4-a_4)}\sum_{k=1}^{n-1}\left(\frac{n-k}{k!}\right)\cdot\left[\frac{\partial^{k-1}f(s_1,s_2,s_3,b_4,x_5,\dots,x_m)}{\partial x_4^{k-1}}(x_4-b_4)^k\right.$$
$$\left.-\frac{\partial^{k-1}f(s_1,s_2,s_3,a_4,x_5,\dots,x_m)}{\partial x_4^{k-1}}(x_4-a_4)^k\right]$$
$$+\frac{1}{(n-1)!\,(b_4-a_4)}\int_{a_4}^{b_4}\int_{s_4}^{x_4}(x_4-t_4)^{n-1}\frac{\partial^n f}{\partial x_4^n}(s_1,s_2,s_3,t_4,x_5,\dots,x_m)\,\mathrm{d}t_4\,\mathrm{d}s_4. \quad (3.85)$$

Combining (3.83) and (3.85) we get

$$f(x_1,x_2,\dots,x_m)=\frac{n^4}{\prod_{j=1}^{4}(b_j-a_j)}\int_{\prod_{j=1}^{4}[a_j,b_j]} f(s_1,...,s_4,x_5,\dots,x_m)\,\mathrm{d}s_1...\mathrm{d}s_4+\sum_{j=1}^{4}T_j, \quad (3.86)$$

where

$$T_4(x_4,x_5,\dots,x_m)=$$
$$\frac{n^3}{\prod_{j=1}^{4}(b_j-a_j)}\sum_{k=1}^{n-1}\frac{n-k}{k!}\int_{\prod_{j=1}^{3}[a_j,b_j]}\left[\frac{\partial^{k-1}f(s_1,s_2,s_3,b_4,x_5,\dots,x_m)}{\partial x_4^{k-1}}(x_4-b_4)^k\right.$$
$$\left.-\frac{\partial^{k-1}f(s_1,s_2,s_3,a_4,x_5,\dots,x_m)}{\partial x_4^{k-1}}(x_4-a_4)^k\right]\mathrm{d}s_1\,\mathrm{d}s_2\,\mathrm{d}s_3$$
$$+\frac{n^3}{(n-1)!\prod_{i=1}^{4}(b_i-a_i)}\int_{\prod_{j=1}^{4}[a_j,b_j]}\int_{s_4}^{x_4}(x_4-t_4)^{n-1}\cdot$$
$$\cdot\frac{\partial^n f}{\partial x_4^n}(s_1,s_2,s_3,t_4,x_5,\dots,x_m)\,\mathrm{d}t_4\,\mathrm{d}s_4\dots\mathrm{d}s_1. \quad (3.87)$$

Etc. doing the same procedure m times, so proving the claim. ☐

Remark 27 *We call for* $i=1,\dots,m$,

$$A_i:=A_i(x_i,\dots,x_m)$$
$$:=\frac{n^{i-1}}{\prod_{j=1}^{i}(b_j-a_j)}\sum_{k=1}^{n-1}\frac{n-k}{k!}\int_{\prod_{j=1}^{i-1}[a_j,b_j]}\left(\frac{\partial^{k-1}f}{\partial x_i^{k-1}}(s_1,\dots,s_{i-1},b_i,x_{i+1},\dots,x_m)(x_i-b_i)^k+\right.$$
$$\left.-\frac{\partial^{k-1}f}{\partial x_i^{k-1}}(s_1,\dots,s_{i-1},a_i,x_{i+1},\dots,x_m)(x_i-a_i)^k\right)\mathrm{d}s_1\dots\mathrm{d}s_{i-1}. \quad (3.88)$$

When $n=1$, then $A_i = 0$ all $i = 1, \ldots, m$. For $i = 1, \ldots, m$, we call

$$B_i := B_i(x_i, \ldots, x_m) := \frac{n^{i-1}}{(n-1)! \prod_{j=1}^{i} (b_j - a_j)} \int_{\prod_{j=1}^{i} [a_j, b_j]} \int_{s_i}^{t_i} (x_i - t_i)^{n-1} \cdot \\ \cdot \frac{\partial^n f}{\partial x_i^n}(s_1, \ldots, s_{i-1}, t_i, x_{i+1}, \ldots, x_m)\, \mathrm{d}t_i\, \mathrm{d}s_i\, \mathrm{d}s_{i-1} \ldots \mathrm{d}s_1, \quad (3.89)$$

so that we have, for all $i = 1, \ldots, m$,

$$T_i = A_i + B_i. \quad (3.90)$$

We set, for $i = 1, \ldots, m$,

$$K(t_i, x_i) := \begin{cases} t_i - a_i, & a_i \le t_i \le x_i, \\ t_i - b_i, & x_i < t_i \le b_i, \end{cases} \quad (3.91)$$

Call, for $i = 1, \ldots, m$,

$$g^{(n)}(t_i) := \frac{\partial^n f}{\partial x_i^n}\left(s_1, s_2, s_3, \ldots, s_{i-1}, t_i, x_{i+1}, \ldots, x_m\right). \quad (3.92)$$

Then as in [8] one can prove that, for $i = 1, \ldots, m$,

$$\left\| \int_{a_i}^{b_i} \int_{s_i}^{t_i} (x_i - t_i)^{n-1} g^{(n)}(t_i)\, \mathrm{d}t_i\, \mathrm{d}s_i \right\| \le \int_{a_i}^{b_i} |x_i - t_i|^{n-1} |K(t_i, x_i)| \cdot \left\| g^{(n)}(t_i) \right\| \mathrm{d}t_i. \quad (3.93)$$

The last gives, for $i = 1, \ldots, m$,

$$\|B_i\| \le \frac{n^{i-1}}{(n-1)! \prod_{j=1}^{i} (b_j - a_j)} \int_{\prod_{j=1}^{i} [a_j, b_j]} |x_i - s_i|^{n-1} |K(s_i, x_i)| \cdot \\ \cdot \left\| \frac{\partial^n f}{\partial x_i^n}(s_1, \ldots, s_{i-1}, s_i, x_{i+1}, \ldots, x_m) \right\| \mathrm{d}s_i \ldots \mathrm{d}s_1, \quad (3.94)$$

Thus by (3.76) we get

$$E_f(x_1, \ldots, x_m) := f(x_1, \ldots, x_m) - \frac{n^m}{\prod_{j=1}^{m} (b_j - a_j)} \int_{\prod_{j=1}^{m} [a_j, b_j]} f(s_1, \ldots, s_m)\, \mathrm{d}s_1 \ldots \mathrm{d}s_m - \sum_{i=1}^{m} A_i \\ = \sum_{i=1}^{m} B_i. \quad (3.95)$$

Hence

$$\|E_f(x_1, \ldots, x_n)\| \le \sum_{i=1}^{m} \|B_i\|. \quad (3.96)$$

Next we estimate E_f via some Ostrowski type inequalities.

Theorem 28 *Assume all as in Theorem 26. Then,* $\forall\,(x_1,\ldots,x_m)\in\prod_{j=1}^{m}[a_j,b_j]:$

$$\|E_f(x_1,\ldots,x_m)\|\leq$$
$$\frac{1}{(n-1)!}\sum_{i=1}^{m}\left\|\left\|\frac{\partial^n f}{\partial x_i^n}\left(\overbrace{\cdots}^{-i-},x_{i+1},\ldots,x_m\right)\right\|\right\|_\infty\frac{n^{i-1}}{b_i-a_i}\left\{\frac{b_i(b_i-x_i)^n-a_i(x_i-a_i)^n}{n}+\right.$$
$$\left.+\sum_{\lambda=0}^{n-1}\binom{n-1}{\lambda}(-1)^{n-\lambda-1}\left[\frac{(n+3)\,x_i^{n+1}}{(\lambda+2)(n-\lambda+1)}-\left(\frac{x_i^\lambda a_i^{n-\lambda+1}}{n-\lambda+1}+\frac{x_i^{n-\lambda-1}b_i^{\lambda+2}}{\lambda+2}\right)\right]\right\}. \tag{3.97}$$

Proof. We observe by (3.94) that

$$\|B_i\|\leq\frac{n^{i-1}}{(n-1)!\prod_{j=1}^{i}(b_j-a_j)}\int_{\prod_{j=1}^{i}[a_j,b_j]}|x_i-s_i|^{n-1}|K(s_i,x_i)|\cdot$$
$$\cdot\left\|\frac{\partial^n f}{\partial x_i^n}(s_1,s_2,s_3,\ldots,s_i,x_{i+1},\ldots,x_m)\right\|\,ds_1\ldots ds_i$$
$$\leq\frac{n^{i-1}}{(n-1)!(b_i-a_i)}\left\|\left\|\frac{\partial^n f}{\partial x_i^n}(\overbrace{\cdots}^{-i-},x_{i+1},\ldots,x_m)\right\|\right\|_\infty\cdot$$
$$\cdot\int_{a_i}^{b_i}|x_i-s_i|^{n-1}|K(s_i,x_i)|\,ds_i=:(*). \tag{3.98}$$

We find that

$$\int_{a_i}^{b_i}|x_i-s_i|^{n-1}|K(s_i,x_i)|\,ds_i$$
$$=\int_{a_i}^{x_i}(x_i-s_i)^{n-1}(s_i-a_i)\,ds_i+\int_{x_i}^{b_i}(s_i-x_i)^{n-1}(b_i-s_i)\,ds_i$$
$$=\frac{b_i(b_i-x_i)^n-a_i(x_i-a_i)^n}{n}$$
$$+\sum_{\lambda=0}^{n-1}\binom{n-1}{\lambda}(-1)^{n-\lambda-1}\left[\frac{(n+3)\,x_i^{n+1}}{(\lambda+2)(n-\lambda+1)}-\left(\frac{x_i^\lambda a_i^{n-\lambda+1}}{n-\lambda+1}+\frac{x_i^{n-\lambda-1}b_i^{\lambda+2}}{\lambda+2}\right)\right]=:\gamma_i. \tag{3.99}$$

Therefore we obtain

$$(*)\leq\frac{n^{i-1}}{(n-1)!(b_i-a_i)}\left\|\left\|\frac{\partial^n f}{\partial x_i^n}\left(\overbrace{\cdots}^{-i-},x_{i+1},\ldots,x_m\right)\right\|\right\|_\infty\gamma_i. \tag{3.100}$$

Finally we get

$$\|E_f(x_1,\ldots,x_m)\|\leq\sum_{i=1}^{m}\frac{n^{i-1}}{(n-1)!(b_i-a_i)}\left\|\left\|\frac{\partial^n f}{\partial x_i^n}\left(\overbrace{\cdots}^{-i-},x_{i+1},\ldots,x_m\right)\right\|\right\|_\infty\gamma_i, \tag{3.101}$$

proving the claim. ☐

We continue with

Theorem 29 *Assume all as in Theorem 26. Then,* $\forall\left(x_1,\ldots,x_m\right)\in\prod_{j=1}^{m}\left[a_j,b_j\right]$:

$$\left\|E_f\left(x_1,x_2\ldots,x_m\right)\right\|\leq\frac{1}{(n-1)!}\sum_{i=1}^{m}\left\|\left\|\frac{\partial^n f}{\partial x_i^n}\right\|\right\|_\infty\frac{n^{i-1}}{b_i-a_i}\left\{\frac{b_i\left(b_i-x_i\right)^n-a_i\left(x_i-a_i\right)^n}{n}\right.$$

$$\left.+\sum_{\lambda=0}^{n-1}\binom{n-1}{\lambda}(-1)^{n-\lambda-1}\left[\frac{(n+3)\,x_i^{n+1}}{(\lambda+2)\,(n-\lambda+1)}-\left(\frac{x_i^\lambda a_i^{n-\lambda+1}}{n-\lambda+1}+\frac{x_i^{n-\lambda-1}b_i^{\lambda+2}}{\lambda+2}\right)\right]\right\}. \quad (3.102)$$

Proof. From Theorem 28. □

Next we give

Theorem 30 *Assume all as in Theorem 26, and let* $p,q>1$ *such that* $\frac{1}{p}+\frac{1}{q}=1$. *Then* $\forall\left(x_1,\ldots,x_m\right)\in\prod_{j=1}^{m}\left[a_j,b_j\right]$:

$$\left\|E_f\left(x_1,\ldots,x_m\right)\right\|\leq\frac{1}{(n-1)!}\sum_{i=1}^{m}\frac{n^{i-1}}{\left(b_i-a_i\right)\left(\prod_{j=1}^{i-1}\left(b_j-a_j\right)\right)^{\frac{1}{p}}}\cdot$$

$$\cdot\left\{\left[\left(x_i-a_i\right)^{nq+1}+\left(b_i-x_i\right)^{nq+1}\right]B((n-1)q+1,q+1)\right\}^{\frac{1}{q}}\left\|\left\|\frac{\partial^n f}{\partial x_i^n}(\overbrace{\cdots}^{-i-},x_{i+1},\ldots,x_m)\right\|\right\|_p. \quad (3.103)$$

Proof. We notice that

$$\left\|B_i\right\|\leq\frac{n^{i-1}}{(n-1)!\prod_{j=1}^{i}\left(b_j-a_j\right)}\left\|\left\|\frac{\partial^n f}{\partial x_i^n}(\overbrace{\cdots}^{-i-},x_{i+1},\ldots,x_m)\right\|\right\|_p\times$$

$$\times\left(\int_{a_i}^{b_i}\left|x_i-s_i\right|^{q(n-1)}\left|K(s_i,x_i)\right|^q\,\mathrm{d}s_i\right)^{\frac{1}{q}}\left(\prod_{j=1}^{i-1}\left(b_j-a_j\right)\right)^{\frac{1}{q}}$$

(by [14, p. 256])

$$=\frac{n^{i-1}}{(n-1)!\left(b_i-a_i\right)\left(\prod_{j=1}^{i-1}\left(b_j-a_j\right)\right)^{\frac{1}{p}}}\cdot$$

$$\cdot\left\{\left[\left(x_i-a_i\right)^{nq+1}+\left(b_i-x_i\right)^{nq+1}\right]B((n-1)q+1,q+1)\right\}^{\frac{1}{q}}\left\|\left\|\frac{\partial^n f}{\partial x_i^n}(\overbrace{\cdots}^{-i-},x_{i+1},\ldots,x_m)\right\|\right\|_p,$$

proving the claim. □

We also present

Theorem 31 *Assume all as in Theorem 26. Then,* $\forall\left(x_1,\ldots,x_m\right)\in\prod_{j=1}^{m}\left[a_j,b_j\right]$:

$$\left\|E_f\left(x_1,\ldots,x_m\right)\right\| \leq \frac{1}{(n-1)!}\sum_{i=1}^{m}\frac{n^{i-1}}{\prod_{j=1}^{i}\left(b_j-a_j\right)}\left(\max\left(x_i-a_i,b_i-x_i\right)\right)^n \cdot \\ \cdot\left\|\left\|\frac{\partial^n f}{\partial x_i^n}(\overbrace{\ldots}^{-i-},x_{i+1},\ldots,x_m)\right\|\right\|_1 . \quad (3.104)$$

Proof. We see that

$$\left\|B_i\right\| \leq \frac{n^{i-1}}{(n-1)!\prod_{j=1}^{i}\left(b_j-a_j\right)}\sup_{s_i\in[a_i,b_i]}\left(\left|x_i-s_i\right|^{n-1}\left|K(s_i,x_i)\right|\right)\left\|\left\|\frac{\partial^n f}{\partial x_i^n}(\overbrace{\ldots}^{-i-},x_{i+1},\ldots,x_m)\right\|\right\|_1$$

$$\leq \frac{n^{i-1}}{(n-1)!\prod_{j=1}^{i}\left(b_j-a_j\right)}\left(\max\left(x_i-a_i,b_i-x_i\right)\right)^n\left\|\left\|\frac{\partial^n f}{\partial x_i^n}\left(\overbrace{\ldots}^{-i-},x_{i+1},\ldots,x_m\right)\right\|\right\|_1, \quad (3.105)$$

proving the claim. ☐

We give

Corollary 32 *Assume all as in Theorem 26. Then, for* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$,
$\forall\left(x_1,\ldots,x_m\right)\in\prod_{j=1}^{m}\left[a_j,b_j\right]$:

$$\left\|E_f\left(x_1,\ldots,x_m\right)\right\| \leq \min\left\{R.H.S.(3.97),R.H.S.(3.103),R.H.S.(3.104)\right\}. \quad (3.106)$$

Proof. By (3.97), (3.103) and (3.104). ☐

We further give

Corollary 33 *Let* $(X,\|\cdot\|)$ *be a Banach space and* $f\in C^1\left(\left[a_1,b_1\right]\times\left[a_2,b_2\right],X\right)$. *Let* $(x_1,x_2)\in\left[a_1,b_1\right]\times\left[a_2,b_2\right]$. *Then*

$$f(x_1,x_2)=\frac{1}{\left(b_1-a_1\right)\left(b_2-a_2\right)}\int_{[a_1,b_1]\times[a_2,b_2]}f(s_1,s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2+T_1+T_2, \quad (3.107)$$

where

$$T_1=T_1(x_1,x_2)=\frac{1}{\left(b_1-a_1\right)}\int_{a_1}^{b_1}K(s_1,x_1)\frac{\partial f}{\partial x_1}(s_1,s_2)\,\mathrm{d}s_1=B_1(x_1,x_2), \quad (3.108)$$

$$T_2=T_2(x_2)=\frac{1}{\left(b_1-a_1\right)\left(b_2-a_2\right)}\int_{a_1}^{b_1}\int_{a_2}^{b_2}K(s_2,x_2)\frac{\partial f}{\partial x_2}(s_1,s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2=B_2(x_2). \quad (3.109)$$

Proof. By Theorem 26. □

We need

Remark 34 *Denote here*

$$E_f(x_1, x_2) = f(x_1, x_2) - \frac{1}{(b_1 - a_1)(b_2 - a_2)} \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(s_1, s_2)\, \mathrm{d}s_1\, \mathrm{d}s_2. \tag{3.110}$$

Hence

$$\|E_f(x_1, x_2)\| \le \|B_1\| + \|B_2\|. \tag{3.111}$$

We give the following special Ostrowski type inequalities.

Corollary 35 *Assume all as in Corollary 33. Then,* $\forall\ (x_1, x_2) \in [a_1, b_1] \times [a_2, b_2]$:

$$\begin{aligned} \|E_f(x_1, x_2)\| &= \left\| f(x_1, x_2) - \frac{1}{(b_1 - a_1)(b_2 - a_2)} \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(s_1, s_2)\, \mathrm{d}s_1\, \mathrm{d}s_2 \right\| \\ &\le \left\| \left\| \frac{\partial f}{\partial x_1}(\cdot, x_2) \right\| \right\|_{\infty, [a_1, b_1]} \frac{\left[a_1^2 + b_1^2 - x_1(a_1 + b_1)\right]}{(b_1 - a_1)} \\ &\quad + \left\| \left\| \frac{\partial f}{\partial x_2}(\cdot, \cdot) \right\| \right\|_{\infty, [a_1, b_1] \times [a_2, b_2]} \frac{\left[a_2^2 + b_2^2 - x_2(a_2 + b_2)\right]}{(b_2 - a_2)}. \end{aligned} \tag{3.112}$$

Proof. By Theorem 28. □

We continue with

Corollary 36 *Assume all as in Corollary 33.* $\forall\ (x_1, x_2) \in [a_1, b_1] \times [a_2, b_2]$:

$$\begin{aligned} |E_f(x_1, x_2)| \le\ &\max \left\{ \left\| \left\| \frac{\partial f}{\partial x_1} \right\| \right\|_{\infty}, \left\| \left\| \frac{\partial f}{\partial x_2} \right\| \right\|_{\infty} \right\} \times \\ &\times \left\{ \left(\frac{a_1^2 + b_1^2 - x_1(a_1 + b_1)}{b_1 - a_1} \right) + \left(\frac{a_2^2 + b_2^2 - x_2(a_2 + b_2)}{b_2 - a_2} \right) \right\}. \end{aligned} \tag{3.113}$$

Proof. By (3.112). □

Next we have

Corollary 37 *Assume all as in Corollary 33. Then,* $\forall\ (x_1, x_2) \in [a_1, b_1] \times [a_2, b_2]$:

$$\begin{aligned} &\left\| f(x_1, x_2) - \frac{1}{(b_1 - a_1)(b_2 - a_2)} \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(s_1, s_2)\, \mathrm{d}s_1\, \mathrm{d}s_2 \right\| \\ &\le \frac{1}{(b_1 - a_1)} \sqrt{\frac{(x_1 - a_1)^3 + (b_1 - x_1)^3}{3}} \left\| \left\| \frac{\partial f}{\partial x_1}(\cdot, x_2) \right\| \right\|_{2, [a_1, b_1]} \\ &\quad + \frac{1}{(b_2 - a_2)\sqrt{b_1 - a_1}} \sqrt{\frac{(x_2 - a_2)^3 + (b_2 - x_2)^3}{3}} \left\| \left\| \frac{\partial f}{\partial x_2}(\cdot, \cdot) \right\| \right\|_{2, [a_1, b_1] \times [a_2, b_2]}. \end{aligned} \tag{3.114}$$

Proof. By (3.103). □

We present

Corollary 38 *Assume all as in Corollary 33. Then,* $\forall\,(x_1,x_2)\in[a_1,b_1]\times[a_2,b_2]:$

$$\left\|f(x_1,x_2)-\frac{1}{(b_1-a_1)(b_2-a_2)}\int_{a_1}^{b_1}\int_{a_2}^{b_2}f(s_1,s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2\right\| \\ \leq\frac{1}{(b_1-a_1)}\max(x_1-a_1,b_1-x_1)\left\|\left\|\frac{\partial f}{\partial x_1}(\cdot,x_2)\right\|\right\|_{1,[a_1,b_1]} \\ +\frac{1}{(b_1-a_1)(b_2-a_2)}\max(x_2-a_2,b_2-x_2)\left\|\left\|\frac{\partial f}{\partial x_2}(\cdot,\cdot)\right\|\right\|_{1,[a_1,b_1]\times[a_2,b_2]}. \tag{3.115}$$

Proof. By (3.104). □

We end Ostrowski type inequality applications for $n=1$, $m=2$ with

Corollary 39 *Assume all as in Corollary 33. Then,* $\forall\,(x_1,x_2)\in[a_1,b_1]\times[a_2,b_2]:$

$$\left\|f(x_1,x_2)-\frac{1}{(b_1-a_1)(b_2-a_2)}\int_{a_1}^{b_1}\int_{a_2}^{b_2}f(s_1,s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2\right\| \\ \leq\min\{R.H.S.(3.112),R.H.S.(3.114),R.H.S.(3.115)\}. \tag{3.116}$$

Corollary 40 *Let* $(X,\|\cdot\|)$ *be a Banach space and* $f\in C^2\left(\prod_{j=1}^{3}[a_j,b_j],X\right)$. *Let also* $(x_1,x_2,x_3)\in\prod_{j=1}^{3}[a_j,b_j]$. *Then*

$$f(x_1,x_2,x_3)=\frac{8}{\prod_{j=1}^{3}(b_j-a_j)}\int_{\prod_{j=1}^{3}[a_j,b_j]}f(s_1,s_2,s_3)\,\mathrm{d}s_1\,\mathrm{d}s_2\,\mathrm{d}s_3+\sum_{i=1}^{3}T_i. \tag{3.117}$$

Here for $i=1,2,3$ *we have*

$$T_1=T_1(x_1,x_2,x_3)=\frac{1}{(b_1-a_1)}\left\{f(b_1,x_2,x_3)(x_1-b_1)-f(a_1,x_2,x_3)(x_1-a_1)\right. \\ \left.+\int_{a_1}^{b_1}(x_1-s_1)\,K(s_1,x_1)\,\frac{\partial^2 f}{\partial x_1^2}(s_1,x_2,x_3)\,\mathrm{d}s_1\right\}, \tag{3.118}$$

$$T_2=T_2(x_2,x_3) \\ =\frac{2}{(b_1-a_1)(b_2-a_2)}\left\{\int_{a_1}^{b_1}\left(f(s_1,b_2,x_3)(x_2-b_2)-f(s_1,a_2,x_3)(x_2-a_2)\right)\mathrm{d}s_1\right. \\ \left.+\int_{\prod_{j=1}^{2}[a_j,b_j]}(x_2-s_2)\,K(s_2,x_2)\,\frac{\partial^2 f}{\partial x_2^2}(s_1,s_2,x_3)\,\mathrm{d}s_1\,\mathrm{d}s_2\right\}, \tag{3.119}$$

and

$$\begin{aligned} T_3 &= T_3(x_3) \\ &= \frac{4}{\prod\limits_{j=1}^{3}(b_j - a_j)} \left\{ \int_{\prod_{j=1}^{2}[a_j,b_j]} \left(f(s_1,s_2,b_3)(x_3-b_3) - f(s_1,s_2,a_3)(x_3-a_3)\right) \mathrm{d}s_1 \, \mathrm{d}s_2 \right. \\ &\qquad \left. + \int_{\prod_{j=1}^{3}[a_j,b_j]} (x_3 - s_3) \, K(s_3,x_3) \, \frac{\partial^2 f}{\partial x_3^2}(s_1,s_2,s_3) \, \mathrm{d}s_1 \, \mathrm{d}s_2 \, \mathrm{d}s_3 \right\}. \end{aligned} \tag{3.120}$$

Remark 41 *Here we have*

$$A_1 = A_1(x_1,x_2,x_3) = \frac{f(b_1,x_2,x_3)(x_1-b_1) - f(a_1,x_2,x_3)(x_1-a_1)}{b_1-a_1}, \tag{3.121}$$

$$\|B_1\| = \|B_1(x_1,x_2,x_3)\| \le \frac{1}{(b_1-a_1)} \int_{a_1}^{b_1} |x_1 - s_1| \, |K(s_1,x_1)| \left\| \frac{\partial^2 f}{\partial x_1^2}(s_1,x_2,x_3) \right\| \mathrm{d}s_1, \tag{3.122}$$

also

$$\begin{aligned} A_2 &= A_2(x_2,x_3) \\ &= \frac{2}{(b_1-a_1)(b_2-a_2)} \int_{a_1}^{b_1} \left[f(s_1,b_2,x_3)(x_2-b_2) - f(s_1,a_2,x_3)(x_2-a_2)\right] \mathrm{d}s_1, \end{aligned} \tag{3.123}$$

$$\begin{aligned} \|B_2\| &= \|B_2(x_2,x_3)\| \\ &\le \frac{2}{(b_1-a_1)(b_2-a_2)} \int_{\prod_{j=1}^{2}[a_j,b_j]} |x_2 - s_2| \, |K(s_2,x_2)| \left\| \frac{\partial^2 f}{\partial x_2^2}(s_1,s_2,x_3) \right\| \mathrm{d}s_1 ds_2, \end{aligned} \tag{3.124}$$

and

$$\begin{aligned} A_3 &= A_3(x_3) \\ &= \frac{4}{\prod\limits_{j=1}^{3}(b_j - a_j)} \int_{\prod_{j=1}^{2}[a_j,b_j]} \left[f(s_1,s_2,b_3)(x_3-b_3) - f(s_1,s_2,a_3)(x_3-a_3)\right] \mathrm{d}s_1 \, \mathrm{d}s_2, \end{aligned} \tag{3.125}$$

$$\begin{aligned} \|B_3\| &= \|B_3(x_3)\| \\ &= \frac{4}{\prod\limits_{j=1}^{3}(b_j - a_j)} \int_{\prod_{j=1}^{3}[a_j,b_j]} |x_3 - s_3| \, |K(s_3,x_3)| \left\| \frac{\partial^2 f}{\partial x_3^2}(s_1,s_2,s_3) \right\| \mathrm{d}s_1 \, \mathrm{d}s_2 \, \mathrm{d}s_3. \end{aligned} \tag{3.126}$$

Notice here

$$T_i = A_i + B_i, \quad i = 1,2,3. \tag{3.127}$$

We also denote

$$E_f(x_1,x_2,x_3) := f(x_1,x_2,x_3) - \frac{8}{\prod_{j=1}^{3}(b_j-a_j)} \int_{\prod_{j=1}^{3}[a_j,b_j]} f(s_1,s_2,s_3)\,\mathrm{d}s_1\,\mathrm{d}s_2\,\mathrm{d}s_3 - \sum_{i=1}^{3} A_i. \tag{3.128}$$

Hence

$$\|E_f(x_1,x_2,x_3)\| \le \sum_{i=1}^{3} \|B_i\|. \tag{3.129}$$

We give

Corollary 42 *Assume all as in Corollary 40. Then,* $\forall\,(x_1,x_2,x_3) \in \prod_{j=1}^{3}[a_j,b_j]$:

$$\|E_f(x_1,x_2,x_3)\| \le \sum_{i=1}^{3} \left\|\left\|\frac{\partial^2 f}{\partial x_i^2}(\overbrace{\cdots}^{-i-},x_3)\right\|\right\|_\infty \frac{2^{i-1}}{b_i-a_i}\left[\left(\frac{b_i(b_i-x_i)^2 - a_i(x_i-a_i)^2}{2}\right) + \right.$$
$$\left. + \sum_{\lambda=0}^{1}(-1)^{1-\lambda}\left\{\frac{5x_i^3}{(\lambda+2)(3-\lambda)} - \left(\frac{x_i^\lambda a_i^{3-\lambda}}{3-\lambda} + \frac{x_i^{1-\lambda} b_i^{\lambda+2}}{\lambda+2}\right)\right\}\right]. \tag{3.130}$$

Proof. By (3.97). □

We continue with

Corollary 43 *Assume all as in Corollary 40. Then,* $\forall\,(x_1,x_2,x_3) \in \prod_{j=1}^{3}[a_j,b_j]$:

$$\|E_f(x_1,x_2,x_3)\| \le \sum_{i=1}^{3} \left\|\left\|\frac{\partial^2 f}{\partial x_i^2}\right\|\right\|_\infty \frac{2^{i-1}}{b_i-a_i}\left[\frac{b_i(b_i-x_i)^2 - a_i(x_i-a_i)^2}{2}\right.$$
$$\left. + \sum_{\lambda=0}^{1}(-1)^{1-\lambda}\left\{\frac{5x_i^3}{(\lambda+2)(3-\lambda)} - \left(\frac{x_i^\lambda a_i^{3-\lambda}}{3-\lambda} + \frac{x_i^{1-\lambda} b_i^{\lambda+2}}{\lambda+2}\right)\right\}\right]. \tag{3.131}$$

Proof. By (3.102). □

Next we give

Corollary 44 *Assume all as in Corollary 40. Then,* $\forall\,(x_1,x_2,x_3) \in \prod_{j=1}^{3}[a_j,b_j]$:

$$\|E_f(x_1,x_2,x_3)\| \le \sum_{i=1}^{3} \frac{2^{i-1}\sqrt{\frac{(x_i-a_i)^5+(b_i-x_i)^5}{30}}}{(b_i-a_i)\left(\prod_{j=1}^{i-1}(b_j-a_j)\right)^{\frac{1}{2}}} \left\|\left\|\frac{\partial^2 f}{\partial x_i^2}\left(\overbrace{\cdots}^{-i-},x_3\right)\right\|\right\|_2. \tag{3.132}$$

Proof. By (3.103). □

We also present

Corollary 45 *Assume all as in Corollary 40. Then,* $\forall\ (x_1, x_2, x_3) \in \prod_{j=1}^{3} [a_j, b_j]$:

$$\|E_f(x_1, x_2, x_3)\| \leq \sum_{i=1}^{3} \frac{2^{i-1}\ (\max(x_i - a_i, b_i - x_i))^2}{\prod_{j=1}^{i} (b_j - a_j)} \left\| \frac{\partial^2 f}{\partial x_i^2} \left(\overbrace{\cdots}^{-i-}, x_3 \right) \right\|_1 . \tag{3.133}$$

Proof. By (3.104). □

We finish with

Corollary 46 *Assume all as in Corollary 40. Then,* $\forall\ (x_1, x_2, x_3) \in \prod_{j=1}^{3} [a_j, b_j]$,

$$\|E_f(x_1, x_2, x_3)\| \leq \min\{R.H.S.(3.130), R.H.S.(3.132), R.H.S.(3.133)\}. \tag{3.134}$$

Proof. By Corollary 32. □

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